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Central limit theorems for the large-spin asymptotics of quantum spins

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Abstract. We use a generalized form of Dyson's spin wave formalism to prove several central limit theorems for the large-spin asymptotics of quantum spins in a coherent state.

1. Introduction

In statistical mechanics, the thermodynamic limit of an infinite number of interacting particles in the continuum or on a lattice can be taken rigorously using the laws of probability. In a first approximation we are usually interested in the behavior of intensive observables, i.e., observables that grow proportionally to the total number of particles. Taking their thermodynamic limit corresponds to the Law of Large Numbers (LLN) in probability theory. Introducing quantum mechanics at the microscopic level does not change much at the macroscopic level: intensive observables still behave classically in the thermodynamic limit, or, the LLN for quantum systems, in particular, involves a description of the system in terms of classical, i.e., commuting, variables.

The next logical step is to study fluctuations of intensive observables around their mean. The corresponding law in probability theory is the Central Limit Theorem (CLT). Here, quantum effects can survive in the limit of an infinite number of particles, and fluctuations often behave non-classically and have to be modeled using non-commuting variables. Typically, this happens in the presence of a spon-

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taneously broken continuous symmetry, and the macroscopic quantum fluctuations present in such a system are the well-known Goldstone bosons [1]. A general theory of Goldstone bosons using non-commutative central limit theorems was presented in [17].

The study of non-commutative central limit theorems becomes essential in this context, and a general theory was developed in [7, 10, 8]. The main result is that the macroscopic quantum fluctuations can be identified with a representation of the Canonical Commutation Relations (CCR) in a quasi-free state which is determined by the correlations in the microscopic state. The non-commutative central limit theorem can be considered as the first quantum correction to the LLN for these systems.

In addition to the thermodynamic limit (i.e., the large-N limit), also the so-called classical limit results in an asymtotic description of the system by classical random variables. A well-known example is the classical limit of quantum spins, which was treated rigorously in great detail by Lieb [15]. This is the limit of infinite spin: the quantum spin operators are normalized such that, in a suitable sense, they converge to classical spin variables with values in the unit sphere in $\mathbb{R}^3$.

As an example of such a result we mention the classical limit of the free energy, obtained for a large class of models by Lieb [15]. In this case, the classical limit result tells us that after rescaling each spin operator in the Hamiltonian, the partition function of the quantum model will converge to the partition function of the corresponding classical model. One may also consider the limit of large spin in a sequence of states, and compute expectations, in which case one is interested in the classical limit as a result about the distribution of the rescaled spin operators considered as random variables. In this present work we are interested in the latter situation.

In Lieb's treatment of the classical limit an important role is played by the so-called coherent states, which were studied in great detail in [2]. Among all quantum states, the coherent states are those that optimally approximate the idea of a spin pointing in a certain direction in space. Mathematically this is made precise by showing that these states carry no dispersion for the classically rescaled spin operators (see Proposition 2.1). Again, we see that the classical limit for spins, which is similar to a LLN in probability, results in classical random variables. The role of N is played by the magnitude of the spin, which we will denote by J. It is now reasonable to expect that in such states in which the 'intensive' (scaled) observables become dispersionless in the limit, a central limit theorem should hold for the fluctuations of these observables around their expected mean. The subject of this paper is then the formulation and proof of various central limit theorems for these fluctuations, which are represented by operators on a Hilbert space. The class of states we consider are products of so-called coherent states for the spins.

In Section 2, which also contains the mathematical setup including the definition and basic properties of the coherent states, we formulate a first non-commutative Central Limit Theorem (CLT) for large J using the techniques of Goderis, Verbeure, and Vets [7, 10, 8]. In a rather different setup, a similar result was previously given by Michoel and Verbeure in [16]. The proof of this CLT and some auxiliary results are given in Appendix A.

The main purpose of this paper is to strengthen the results of Section 2, which could be called standard, by taking advantage of the additional structure that is present in models with a high dimensional representation of the $SU(2)$ commutation relations. More precisely, we want to use Dyson's spin wave formalism [5, 6], which can be briefly described as follows. The 'all spin up' state is used as a reference state in spin Hilbert space. It plays the role of the vacuum state in the sense that there is a formal analogy between lowering the spin in this state, and creating a boson particle in a Fock vacuum state. Dyson made this analogy precise by defining a unitary equivalence between the spin Hilbert space and a subspace of Fock space (the subspace with no more particles present than the size of the spin). Under this equivalence, the spin lowering operator, divided by the square root of the size of the spin, becomes the boson creation operator up to a correction that is a function of the number operator that formally goes to one in the large-spin limit. After the trivial observation that every coherent state is the 'all spin up' state for the spin in the defining direction of the coherent state, we see that Dyson's formalism can be used to study the coherent states of our situation. The spin wave formalism is discussed in detail in Section 3.

The scaling with the square root of the spin in Dyson's equivalence between the spin lowering and the boson creation operator is precisely the scaling we need in the central limit theorem for the fluctuations of the spin operators. Hence, for all values of the spin, we can write the fluctuation operators as well-defined operators on one and the same Fock space, and it follows (see Section 4), that the limit of infinite spin can be taken as a genuine operator limit. This is where the central limit theorem for spin fluctuations differs fundamentally from the known non-commutative central limit theorems and it makes it possible to prove stronger results. For instance, the usual central limit theorems prove convergence of the characteristic function of linear combinations of fluctuation operators. Using Dyson's spin wave formalism, we obtain convergence of the characteristic function of arbitrary polynomials of fluctuation operators.

In the last section, Section 5, we discuss some applications of this stronger version of the central limit theorem.

The first application (in Section 5.1) is to obtain a rigorous version of "bosonization" for quantum spin systems. This refers to the well-known technique in physics that the low-energy excitations, and hence the low-temperature behavior, of certain quantum spin systems can be well approximated using a boson or spin wave approximation. This idea can be made mathematically precise for those quantum spin systems which possess a coherent ground state. Indeed, using our central limit theorem we immediately obtain strong convergence of the spin Hamiltonian and the spin dynamics to a quasi-free boson Hamiltonian and dynamics. In [18], we use this convergence to obtain the large-spin asymptotics of the energy spectrum of the anisotropic ferromagnetic Heisenberg chain, thus improving upon earlier results [3, 12]. We also discuss another application of this convergence result, namely to obtain the large-spin asymptotics for the time evolution of the ground state when the dynamics is perturbed by a spin fluctuation.

In a second application (Section 5.2), we apply the theorem to the study of non-commuting fluctuation operators for N independent copies of quantum random variables. A central limit theorem for the characteristic function of polynomials of such fluctuation operators, analogous to the one we obtain in Section 4, has appeared in the literature in the form of a conjecture by Kuperberg [13]. The special case of this conjecture for expectations computed in a tracial state, was proved in [14] and establishes an interesting result about the distribution of the shape of a random word. This is an example of a quantum CLT used to prove results in classical probability theory, in this case generalizing earlier work of Johansson [11].

The situation with a tracial state can be regarded as intermediate between classical and quantum probability theory, because the cyclicity of the trace implies that although the microscopic variables do not commute, the limiting fluctuations are classical Gaussian random variables. We can apply our results to obtain the first example of a non-tracial state for which Kuperberg's conjecture holds. We show that a system of N independent spin-$\frac{1}{2}$ particles, each in a coherent state, can be identified with one spin-$\frac{N}{2}$ particle in the corresponding coherent state. Therefore, our general results can be applied directly and we obtain a fully non-commutative system in which the conjecture is valid.

2. Mathematical setup and preliminary results

We consider quantum spin systems on a finite or infinite lattice $\mathfrak{L}$. At each site $x \in \mathfrak{L}$ we have a spin-J degree of freedom ($J \in \frac{1}{2}\mathbb{N}_0$), i.e., a $(2J+1)$-dimensional irreducible representation of $SU(2)$, and we denote with S_x^i the corresponding spin-J matrices,

$$\begin{aligned}[S_x^i, S_y^j] &= i\delta_{x,y}\varepsilon_{ijk}S_x^k \\ S_x \cdot S_x &= (S_x^1)^2 + (S_x^2)^2 + (S_x^3)^2 = J(J+1)\end{aligned}$$

We will also use the spin raising and lowering operators: S_x^+ and S_x^-, $S_x^\pm = S_x^1 \pm i S_x^2$.

The local Hilbert spaces associated to each finite subset Λ of the lattice are therefore

$$\mathcal{H}_{J,\Lambda} = \bigotimes_{x\in\Lambda}\left(\mathbb{C}^{2J+1}\right)_x$$

and the algebra of observables is

$$\mathfrak{A}_{J,\Lambda} = \bigotimes_{x\in\Lambda}\left(\mathbb{M}_{2J+1}(\mathbb{C})\right)_x$$

For $\Lambda' \subset \Lambda$, $\mathfrak{A}_{J,\Lambda'}$ can be considered as a subalgebra of $\mathfrak{A}_{J,\Lambda}$ in a natural way. With this in mind, the union

$$\mathfrak{A}_{J,\mathrm{loc}} = \bigcup_{\Lambda\subset\mathfrak{L}} \mathfrak{A}_{\Lambda,J}$$

is the algebra of local observables, and its closure is the quasi-local algebra of observables $\mathfrak{A}_J$ of the spin system on $\mathfrak{L}$. For a finite lattice $\mathfrak{L}$ this is of course just the local algebra specified before.

As a basis for the local Hilbert spaces we take the standard basis which is diagonal for the S_x^3 operators:

$$S_x^3|m_x\rangle = m_x|m_x\rangle, \quad m_x = -J, -J+1, \ldots, J$$
$$S_x^{\pm}|m_x\rangle = \sqrt{J(J+1) - m_x(m_x \pm 1)}|m_x \pm 1\rangle$$

A coherent spin state [2, 15] at site x is specified by a unit vector in $\mathbb{R}^3$ (i.e., a *classical* spin) $u_x = (\theta_x, \varphi_x)$, $0 \leq \theta_x \leq \pi$, $0 \leq \varphi_x \leq 2\pi$, and defined by

$$\begin{aligned}|(\theta_x, \varphi_x)\rangle &= e^{\frac{1}{2}\theta_x(S_x^- e^{i\varphi_x} - S_x^+ e^{-i\varphi_x})}|J\rangle \\ &= \sum_{m_x=-J}^{J} \binom{2J}{J - m_x}^{1/2} (\cos\tfrac{1}{2}\theta_x)^{J+m_x} (\sin\tfrac{1}{2}\theta_x)^{J-m_x} e^{i(J-m_x)\varphi_x} \; |m_x\rangle\end{aligned}$$

The states of the spin system that we will consider are tensor products of coherent states:

$$\omega(A) = \bigotimes_{x \in \mathfrak{L}} \langle(\theta_x, \varphi_x)|A|(\theta_x, \varphi_x)\rangle$$

which is well-defined for all $A \in \mathfrak{A}_J$. The collection of unit vectors $\{u_x = (\theta_x, \varphi_x)\}_{x \in \mathfrak{L}}$ defining the state is completely arbitrary. The GNS Hilbert space of ω is the incomplete tensor product Hilbert space

$$\mathcal{H}_J = \overline{\bigcup_{\Lambda \subset \mathfrak{L}} \Big(\big[\bigotimes_{x \in \Lambda} \mathbb{C}^{2J+1}\big] \otimes \big[\bigotimes_{y \in \mathfrak{L} \setminus \Lambda} \Omega_y\big]\Big)}$$

where Ω_y is short-hand for $\Omega_y = |(\theta_y, \varphi_y)\rangle$, and the generating vector is

$$\Omega_J = \bigotimes_{x \in \mathfrak{L}} \Omega_x$$

Because of the simplicity of the GNS representation we will not need to distinguish between an observable and its representative in $\mathcal{B}(\mathcal{H}_J)$.

A particular type of quasi-local observables that we are interested in are those of the form

$$\sum_{x \in \mathfrak{L}} v_x \cdot S_x, \quad v_x \in \mathbb{R}^3 \tag{1}$$

In order that functions, such as polynomials and trigonometric functions, of these observables are well-defined, we need summability conditions on the set of v_x. Consider $v = \{v_x \in \mathbb{R}^d\}_{x \in \mathfrak{L}}$, and define the p-norm ($p \geq 1$) by

$$|v|_p := \Big(\sum_{x \in \mathfrak{L}} |v_x|^p\Big)^{1/p}$$

where $|v_x|$ is the length of v_x. The space of all $\mathbb{R}^d$-valued summable sequences (i.e., $|v|_1 < \infty$) on $\mathfrak{L}$ will be denoted $\ell_d^1(\mathfrak{L})$, and the space of all $\mathbb{R}^d$-valued square summable sequences will be denoted $\ell_d^2(\mathfrak{L})$. For $d = 2$ we get the standard complex valued (square) summable sequences, and for these spaces the subscript d is omitted. Note that if $|v|_{p_0} < \infty$ for some $p_0 \geq 1$, then for $p \geq p_0$, $|v|_p \leq |v|_{p_0} < \infty$ as well.

In [2, 15], a number of generating functions are derived for coherent states. The most important one for us is the characteristic function for observables of the type (1):

$$\omega\big(e^{i\sum_x v_x\cdot S_x}\big) = \prod_{x\in\mathfrak{L}} \Big\{\cos(\frac{1}{2}|v_x|) + i\frac{v_x\cdot u_x}{|v_x|}\sin(\frac{1}{2}|v_x|)\Big\}^{2J} \tag{2}$$

It follows immediately that

$$\omega\big(\sum_x v_x\cdot S_x\big) = J\sum_x v_x\cdot u_x$$

The classical limit of a quantum spin system is obtained by normalizing each spin operator by J, and then taking the limit J to ∞. The quantum spin variables then converge to classical spins taking values in the unit sphere in $\mathbb{R}^3$. There are various precise mathematical statements that can express this. The most common are results formulated as convergence of the free energy of a quantum spin system to the free energy of the corresponding classical spin system [15]. In the present paper, we do not specify a Hamiltonian, but consider the characteristic functions of the spin variables in states that are products of coherent states. The following result, proved in Appendix A as an easy consequence of eq. (2), is a law of large numbers for the quantum spins.

Proposition 2.1 (Classical limit). *For $v \in \ell^1_3(\mathfrak{L})$ and $J \in \frac{1}{2}\mathbb{N}_0$, we have*

$$\Big|\omega\big(e^{\frac{i}{J}\sum_x v_x\cdot S_x}\big) - e^{i\sum_x v_x\cdot u_x}\Big| \leq \frac{1}{J}\exp\Big[|v|_1 + 2\exp\big(|v|_1\big)\Big]$$

More general results for products of characterisitc functions, analogous to Theorem 2.3 below, can also be derived for the classical limit, but the above proposition is sufficient for our needs.

The next step is to study the fluctuations of these observables around their mean and we define fluctuation observables by

$$F_J(v) = \sqrt{\frac{2}{J}}\sum_{x\in\mathfrak{L}}\big[v_x\cdot S_x - \omega(v_x\cdot S_x)\big] \tag{3}$$

Our first result here is about the characteristic function of the fluctuation observables in a coherent state:

Proposition 2.2. *For $v \in \ell^2_3(\mathfrak{L})$ and $J \in \frac{1}{2}\mathbb{N}_0$, we have*

$$\Big|\omega\big(e^{iF_J(v)}\big) - e^{-\frac{1}{2}|\tilde{v}|_2^2}\Big| \leq \frac{1}{J^{1/2}}b(v)$$

where

$$b(v) = \exp\Big[2^{1/2}|v|_2 + 2^{1/2}\exp\big(2^{1/2}|v|_2\big)\Big] \tag{4}$$

The vector $\tilde{v} \in \ell^2(\mathfrak{L})$ in the proposition whose length determines the variance of $F_J(v)$ is defined as follows. At each site x we start with the same standard basis (e_x^1, e_x^2, e_x^3) of $\mathbb{R}^3$. Through the state ω, we are given a unit vector $u_x = (\theta_x, \varphi_x)$ where the spherical coordinates are given w.r.t. the standard basis. We can define a new basis of $\mathbb{R}^3$ (now site dependent) by rotating e_x^3 to u_x, i.e., define

$$f_x^i = \sum_{j=1,2,3} R_{ij}(\theta_x, \varphi_x) e_x^j$$

with $R_{ij}(\theta_x, \varphi_x)$ the rotation matrix

$$R_{ij}(\theta_x, \varphi_x) = \begin{pmatrix} \cos\theta_x \cos\varphi_x & \cos\theta_x \sin\varphi_x & -\sin\theta_x \\ -\sin\varphi_x & \cos\varphi_x & 0 \\ \sin\theta_x \cos\varphi_x & \sin\theta_x \sin\varphi_x & -\cos\theta_x \end{pmatrix}$$

The vectors f_x^1 and f_x^2 span the plane orthogonal to u_x which can be identified with the tangent plane to the unit sphere at u_x. We define $\tilde{v}_x \in \mathbb{R}^2$ to be the projection of v_x onto this plane.

To see that this is really the variance, we can compute directly from eq. (2) that

$$\omega\big(F_J(v)F_J(w)\big) = \sum_{x\in\mathfrak{L}} \big[v_x \cdot w_x - (v_x \cdot u_x)(w_x \cdot u_x) + i(v_x \times w_x) \cdot u_x\big] \quad (5)$$

and it follows that indeed

$$\omega\big(F_J(v)^2\big) = |\tilde{v}|_2^2$$

The proof of Proposition 2.2 is given in Appendix A as well.

We can continue along the lines of [7, 8, 10] and show that in the limit $J \to \infty$ the system of fluctuations is given by a representation of the canonical commutation relations (CCR). The most general theorem in this context is:

Theorem 2.3. *For $n \in \mathbb{N}_0$, $v_1, \ldots, v_n \in \ell_3^2(\mathfrak{L})$ and $J \in \frac{1}{2}\mathbb{N}_0$,*

$$\left|\omega\Big(\prod_{j=1}^n e^{iF_J(v_j)}\Big) - \tilde{\omega}\Big(\prod_{j=1}^n W(\tilde{v}_j)\Big)\right| \le \frac{1}{J^{1/2}}\left\{b\Big(\sum_{j=1}^n v_j\Big) + \sum_{j=1}^{n-1} a\Big(v_j, \sum_{k=j+1}^n v_k\Big)\right\} \quad (6)$$

where it is understood that the second sum is zero for $n = 1$, $b(v)$ is given in (4) *and $a(v, w)$ is given by*

$$a(v, w) = \frac{1}{3}|v|_2|w|_2(|v|_2 + |w|_2) + \sqrt{2}\exp\Big[\frac{1}{2}|v|_2|w|_2 + \exp\big(|v|_2|w|_2\big)\Big]$$

In this theorem, the $\tilde{v}_j \in \ell^2(\mathfrak{L})$ are again defined by projecting onto the tangent planes at the different u_x; the $W(\tilde{v})$ are the Weyl operators generating the CCR algebra $\mathsf{CCR}(\ell^2(\mathfrak{L}), \sigma)$, and σ is the symplectic form associated with the standard inner product in $\ell^2(\mathfrak{L})$, i.e.,

$$\sigma(\tilde{v}, \tilde{w}) = 2\mathfrak{Im}\,\langle\tilde{v}, \tilde{w}\rangle = 2\mathfrak{Im} \sum_{x\in\mathfrak{L}} \overline{\tilde{v}}_x \tilde{w}_x$$

and $\tilde{\omega}$ is the quasi-free Fock state defined by

$$\tilde{\omega}\big(W(\tilde{v})\big) = e^{-\frac{1}{2}\langle\tilde{v},\tilde{v}\rangle} = e^{-\frac{1}{2}|\tilde{v}|_2^2}$$

The Weyl operators satisfy the well-known canonical commutation relations:

$$W(\tilde{v})W(\tilde{w}) = e^{-\frac{i}{2}\sigma(\tilde{v},\tilde{w})}W(\tilde{v}+\tilde{w})$$

The proof of Theorem 2.3 requires a Baker-Campbell-Hausdorff-type formula (BCH formula) that shows that the $e^{iF_J(v)}$ approximate these commutation relations in a suitable sense, and then proceeds from Proposition 2.2 through a standard induction argument. For convenience of the reader, proofs are included in Appendix A.

The proofs of both Proposition 2.1 and 2.2 and Theorem 2.3 rely directly on the use of the generating function (2). In order to prove convergence of moments or convergence of the characteristic function of polynomials of the $F_J(v)$, Dyson's spin wave formalism will prove much more convenient.

3. Dyson's spin wave formalism for coherent states

In his famous papers [5, 6], Dyson introduces a formalism for studying rigorously bosonization in the Heisenberg ferromagnet. The main idea is to take the ferromagnetic ground state $\otimes_{x\in\mathfrak{L}}|J\rangle$ as a reference state, and to identify the lowering of spins in this state with the creation of boson particles in a Fock state. This allows the Hamiltonian to be written as an operator on Fock space, and it is argued that the leading terms at low temperature are the ones linear and quadratic in the creation and annihilation operators, thus obtaining an exactly solvable system. This approximation corresponds to taking a large J limit much as we want to do here. A nice exposition of Dyson's formalism in a more modern language is in [22], and a rigorous theorem about convergence of the free energy within this formalism is in [4].

Upon inspection it is clear that the same formalism can also be used if the reference state is a general product state of coherent states instead of the purely ferromagnetic state $\otimes_x|J\rangle$. This was used in [18] to prove that the low-energy excitations of interface ground states of the 1-dimensional XXZ chain are given, in the large J limit, by a quadratic boson Hamiltonian describing particles hopping on the lattice under the influence of an external potential centred around the interface. In this paper we use the formalism in an analogous way to prove several central limit theorems for the fluctuation observables $F_J(v)$.

Recall the new basis we defined for every site x using the unit vectors u_x that determine the state ω (see after Proposition 2.2). The spin operators can be rotated likewise, and we find (using, e.g., eqs. (3.9) of [2])

$$\tilde{S}_x^i = U_x S_x^i U_x^* = f_x^i \cdot S_x$$

where

$$U_x = e^{\frac{1}{2}\theta_x(S_x^- e^{i\varphi_x} - S_x^+ e^{-i\varphi_x})}$$

The rotated spin raising and lowering operators are $\tilde{S}_x^{\pm} = \tilde{S}_x^1 \pm i\tilde{S}_x^2$.

The main observation is that in this new basis the state $\otimes_x |(\theta_x, \varphi_x)\rangle$ becomes the usual reference state $\otimes_x |+J\rangle$ for the spin wave formalism:

$$\tilde{S}_x^3 |(\theta_x, \varphi_x)\rangle = \tilde{S}_x^3 U_x |J\rangle = U_x S_x^3 |J\rangle = J|(\theta_x, \varphi_x)\rangle$$

According to our previous notation we denote the components of a vector $v_x \in \mathbb{R}^3$ w.r.t. the standard basis with v_x^i, and w.r.t. the rotated basis with $\tilde{v}_x^i$. We find

$$v_x \cdot S_x = \frac{1}{2}\tilde{v}_x^- \tilde{S}_x^+ + \frac{1}{2}\tilde{v}_x^+ \tilde{S}_x^- + \tilde{v}_x^3 \tilde{S}_x^3$$

where $\tilde{v}_x^\pm = \tilde{v}_x^1 \pm i\tilde{v}_x^2$, and

$$F_J(v) = \sum_{x\in\mathfrak{L}} \tilde{v}_x^- \frac{\tilde{S}_x^+}{(2J)^{1/2}} + \tilde{v}_x^+ \frac{\tilde{S}_x^-}{(2J)^{1/2}} + \Big(\frac{2}{J}\Big)^{1/2} \tilde{v}_x^3 (\tilde{S}_x^3 - J)$$

where we used $\omega(v_x \cdot S_x) = J v_x \cdot u_x = J\tilde{v}_x^3$.

Let

$$\vec{n} = \{n_x \in \mathbb{N}\}_{x\in\mathfrak{L}}$$
$$\mathcal{N}_J = \Big\{\vec{n} \mid \forall x\colon n_x \le 2J, \sum_x n_x < \infty\Big\}$$
$$\varphi_{\vec{n}} = \prod_{x\in\mathfrak{L}} \frac{1}{n_x!} \binom{2J}{n_x}^{-\frac{1}{2}} \big(\tilde{S}_x^-\big)^{n_x} \Omega_J$$

where $\Omega_J = \otimes_x |(\theta_x, \varphi_x)\rangle$ is the GNS vector for the state ω. The set $\{\varphi_{\vec{n}} \mid \vec{n} \in \mathcal{N}_J\}$ is an orthonormal basis for $\mathcal{H}_J$, which will prove to be very useful.

Recall that $\tilde{\omega}$ is the quasi-free Fock state on $\mathsf{CCR}(\ell^2(\mathfrak{L}), \sigma)$. Its GNS Hilbert space is the usual Fock space $\mathcal{F}$ with a vacuum vector $\tilde{\Omega}$, and creation and annihilation operators a_x^* and a_x:

$$[a_x, a_y^*] = \delta_{xy}$$
$$a_x \tilde{\Omega} = 0$$

The representative of $W(\tilde{v})$ is the usual Weyl operator

$$W(\tilde{v}) = e^{iF(\tilde{v})} = e^{i\sum_x \tilde{v}_x^+ a_x^* + \tilde{v}_x^- a_x}$$

where, if $\tilde{v}_x = (\tilde{v}_x^1, \tilde{v}_x^2) \in \mathbb{R}^2$, $\tilde{v}_x^\pm \in \mathbb{C}$ are defined as above. The unbounded operator

$$F(\tilde{v}) = \sum_{x\in\mathfrak{L}} \big[\tilde{v}_x^+ a_x^* + \tilde{v}_x^- a_x\big] \tag{7}$$

is well-defined for $\tilde{v} \in \ell^2(\mathfrak{L})$ and is called the boson field operator.

Now let

$$\mathcal{N} = \Big\{\vec{n} \mid \sum_x n_x < \infty\Big\}$$

$$\tilde{\varphi}_{\vec{n}} = \prod_{x\in\mathfrak{L}} \frac{1}{(n_x)^{1/2}} (a_x^*)^{n_x} \tilde{\Omega}$$

then we have an orthonormal basis $\{\tilde{\varphi}_{\vec{n}} \mid \vec{n} \in \mathcal{N}\}$ of $\mathcal{F}$. Identifying $\varphi_{\vec{n}}$ with $\tilde{\varphi}_{\vec{n}}$, it is clear that the GNS Hilbert spaces, $\mathcal{H}_J$, $J \in \frac{1}{2}\mathbb{N}_0$, can be identified with a nested sequence of subspaces of $\mathcal{F}$, defined for each J as the linear span of all vectors $\tilde{\varphi}_{\vec{n}}$, with $n_x \le 2J$. More precisely, we use the projections $P_{n,x}$ on $\mathcal{F}$ which project onto the first $2n$ boson states at site x, i.e., on the states $\tilde{\varphi}_{\vec{n}}$ with $0 \le n_x \le 2n$, and denote $P_n = \prod_x P_{n,x}$, and find

$$\mathcal{H}_J = P_J \mathcal{F}$$

where $=$ means unitarily equivalent.

Under this equivalence, we find that the spin operators are given by (see [22] for more details)

$$\frac{\tilde{S}_x^-}{(2J)^{1/2}} = P_J a_x^* g_J(x)^{1/2}, \quad \frac{\tilde{S}_x^+}{(2J)^{1/2}} = g_J(x)^{1/2} a_x P_J, \quad J - \tilde{S}_x^3 = P_J a_x^* a_x P_J$$

where

$$g_J(x) = g_J(a_x^* a_x)$$

and

$$g_J(n) = \begin{cases} 1 - \frac{1}{2J} n & n \le 2J \\ 0 & n > 2J \end{cases}$$

Hence

$$F_J(v) = P_J \Big\{ \sum_{x\in\mathfrak{L}} \tilde{v}_x^+ a_x^* g_J(x)^{1/2} + \tilde{v}_x^- g_J(x)^{1/2} a_x - \Big(\frac{2}{J}\Big)^{1/2} \tilde{v}_x^3 a_x^* a_x \Big\} P_J \qquad (8)$$

If we let $J \to \infty$, we have, for fixed n, $g_J(n) \to 1$ and formally $F_J(v) \to F(\tilde{v})$, the boson field operator of eq. (7). To make this into a mathematically precise statement is the subject of the next section.

4. Operator convergence of fluctuation operators

For any $n \in \mathbb{N}$, let $\mathcal{D}_n$ denote the linear span of the vectors

$$F(\tilde{w}_1) \dots F(\tilde{w}_n)\tilde{\Omega} \colon \tilde{w}_1, \dots, \tilde{w}_n \in \ell^2(\mathfrak{L})$$

and define $\mathcal{D}$ by

$$\mathcal{D} = \bigoplus_{n\in\mathbb{N}} \mathcal{D}_n,$$

i.e., $\mathcal{D}$ is the linear span of the vectors $\tilde{\varphi}_{\vec{n}}$, $\vec{n} \in \mathcal{N}$. The set $\mathcal{D}$ consists of entire analytic vectors for $F(\tilde{v})$, $\tilde{v} \in \ell^2(\mathfrak{L})$ (Theorem 4.6 of [20]), i.e., $\psi \in \mathcal{F}$ such that ψ is in the domain of $F(\tilde{v})^k$ for every $k \in \mathbb{N}$ and

$$\sum_{k\geq 0} \frac{t^k}{k!} \| F(\tilde{v})^k \psi \| < \infty \quad (t > 0)$$

and therefore $\mathcal{D}$ is dense in $\mathcal{F}$. It also follows that $\mathcal{D}$ is a core for all $F(\tilde{v})$, $\tilde{v} \in \ell^2(\mathfrak{L})$ as well as for the creation and annihilation operators $a^*(\tilde{v})$ and $a(\tilde{v})$,

$$a^*(\tilde{v}) = [a(\tilde{v})]^* = \sum_{x\in\mathfrak{L}} \tilde{v}_x^+ a_x^*$$

Clearly, any function of the number operators, $n_x = a_x^* a_x$, such as P_J, leaves the spaces $\mathcal{D}_n$ invariant and hence:

Lemma 4.1. *For all $n \in \mathbb{N}$, $v_1, \ldots, v_n \in \ell_3^1(\mathfrak{L})$, and $J \in \frac{1}{2}\mathbb{N}_0$ such that $2J > n$, we have*

$$F_J(v_1) \ldots F_J(v_n)\tilde{\Omega} \in \mathcal{D}_n$$

Recall the following lemma, Lemma 4.5 of [20]:

Lemma 4.2. *For all $n \in \mathbb{N}$, $\psi_n \in \mathcal{D}_n$ and $\tilde{v} \in \ell^2(\mathfrak{L})$, we have*

$$\| F(\tilde{v})\psi_n \| \leq 2|\tilde{v}|_2(n+1)^{1/2} \| \psi_n \|$$

The analogous result for the $F_J(v)$ is:

Lemma 4.3. *For all $n \in \mathbb{N}$, $\psi_n \in \mathcal{D}_n$, $v \in \ell_3^1(\mathfrak{L})$, and $J \in \frac{1}{2}\mathbb{N}_0$ such that $2J > n$, we have*

$$\| F_J(v)\psi_n \| \leq 4|v|_1(n+1)^{1/2} \| \psi_n \|$$

Proof. By the choice $2J > n$ we do not need the projection operators P_J, in other words $\mathcal{D}_n \subset \mathcal{H}_J = P_J\mathcal{F}$. For simplicity, denote

$$A_x = \tilde{v}_x^- g_J(x)^{1/2} a_x$$

We have

$$F_J(v) = \sum_x A_x^* + A_x - \Big(\frac{2}{J}\Big)^{1/2} \tilde{v}_x^3 a_x^* a_x$$

and

$$\| F_J(v)\psi_n \| \leq \sum_x \| (A_x^* + A_x)\psi_n \| + \Big(\frac{2}{J}\Big)^{1/2} \sum_x |\tilde{v}_x^3| \| a_x^* a_x \psi_n \|$$

The second term is bounded by $2^{1/2} J^{-1/2} n |v|_1 \| \psi_n \|$; for the first term we use

$$\| (A_x^* + A_x)\psi_n \|^2 \leq 2 \| (A_x^* A_x)^{1/2} \psi_n \|^2 + 2 \| (A_x A_x^*)^{1/2} \psi_n \|^2$$

On $\mathcal{D}_n$, we have, using $2J > n$ as well,

$$A_x^* A_x = |\tilde{v}_x|^2 a_x^* g_J(x) a_x = |\tilde{v}_x|^2 \Big[n_x \big(1 - \frac{n_x}{2J}\big) + \frac{n_x}{2J} \Big] \leq |\tilde{v}_x|^2 (n+1) \mathbb{1}$$

$$A_x A_x^* = |\tilde{v}_x|^2 g_J(x)^{1/2} a_x a_x^* g_J(x)^{1/2} = |\tilde{v}_x|^2 (n_x + 1)\big(1 - \frac{n_x}{2J}\big) \leq |\tilde{v}_x|^2 (n+1) \mathbb{1}$$

and, using once more $2J > n$,

$$\big\| F_J(v) \psi_n \big\| \leq 2|v|_1 \|\psi_n\| \Big[(n+1)^{1/2} + \frac{n}{(2J)^{1/2}} \Big] \leq 4|v|_1 (n+1)^{1/2} \|\psi_n\| \quad \square$$

Denote with s-lim the strong resolvent operator limit for operators acting on $\mathcal{F}$. A first result about the convergence of fluctuation observables is:

Lemma 4.4. *For $v \in \ell_3^1(\mathfrak{L})$, we have*

$$\operatorname*{s-lim}_{J \to \infty} F_J(v) = F(\tilde{v}) \tag{9}$$

where the limit is taken over any sequence of $J \in \frac{1}{2}\mathbb{N}_0$ tending to ∞. More precisely, for all $n \in \mathbb{N}$, $\psi_n \in \mathcal{D}_n$, and $J \in \frac{1}{2}\mathbb{N}_0$ such that $2J > n+1$, we have

$$\big\| \big[F(\tilde{v}) - F_J(v) \big] \psi_n \big\| \leq 4|v|_1 \frac{n}{(2J)^{1/2}} \big\| \psi_n \big\| \tag{10}$$

Proof. Since $\mathcal{D}$ consists of entire analytic vectors for $F(\tilde{v})$, (9) follows directly from (10), see, e.g., Theorem VIII.25 of [21], so it is sufficient to prove (10). This can be done as in the proof of Lemma 4.3. This time,

$$A_x = \tilde{v}_x^- \big(1 - g_J(x)^{1/2}\big) a_x$$

We have

$$\big\| \big[F(\tilde{v}) - F_J(v) \big] \psi_n \big\| \leq \sum_x \big\| (A_x^* + A_x) \psi_n \big\| + \big(\frac{2}{J}\big)^{1/2} \sum_x |\tilde{v}_x^3| \big\| a_x^* a_x \psi_n \big\|$$

The second term is bounded by $2^{1/2} J^{-1/2} n |v|_1 \|\psi_n\|$; for the first term we use again

$$\big\| (A_x^* + A_x) \psi_n \big\|^2 \leq 2 \big\| (A_x^* A_x)^{1/2} \psi_n \big\|^2 + 2 \big\| (A_x A_x^*)^{1/2} \psi_n \big\|^2$$

These two terms can be bounded using $1 - g_J(x)^{1/2} \leq n_x (2J)^{-1}$ and Lemma 4.2:

$$\begin{aligned}
\big\| (A_x^* A_x)^{1/2} \psi_n \big\|^2 &= \langle \psi_n, A_x^* A_x \psi_n \rangle = |\tilde{v}_x|^2 \langle a_x \psi_n, (1 - g_J(x)^{1/2})^2 a_x \psi_n \rangle \\
&\leq |\tilde{v}_x|^2 \big(\frac{n}{2J}\big)^2 \|a_x \psi_n\|^2 \leq |\tilde{v}_x|^2 \big(\frac{n}{2J}\big)^2 n \|\psi_n\|^2 \\
\big\| (A_x A_x^*)^{1/2} \psi_n \big\|^2 &= \langle \psi_n, A_x A_x^* \psi_n \rangle \\
&= |\tilde{v}_x|^2 \langle \psi_n, (1 - g_J(x)^{1/2}) a_x a_x^* (1 - g_J(x)^{1/2}) \psi_n \rangle \\
&\leq |\tilde{v}_x|^2 \big(\frac{n}{2J}\big)^2 (n+1) \|\psi_n\|^2
\end{aligned}$$

Putting everything together, and using $2J > n+1$, we find

$$\big\| \big[F(\tilde{v}) - F_J(v) \big] \psi_n \big\| \leq 2|v|_1 \Big[\frac{n(n+1)^{1/2}}{2J} + \frac{n}{(2J)^{1/2}} \Big] \|\psi_n\| \leq 4|v|_1 \frac{n}{(2J)^{1/2}} \|\psi_n\|$$

$\square$

We will extend this result to arbitrary selfadjoint polynomials in the fluctuation observables. A selfadjoint polynomial is defined as follows. If $A_1, \dots, A_k$ are selfadjoint elements of a general $\mathcal{C}^*$ or von Neumann algebra $\mathfrak{A}$, we denote with $\mathbb{C}\langle A_1, \dots, A_k\rangle$ the ring of noncommutative polynomials in $A_1, \dots, A_k$. This ring admits a *-involution which conjugates each coefficient and reverses the order of multiplication in each term. A polynomial $p \in \mathbb{C}\langle A_1, \dots, A_k\rangle$ is called a selfadjoint polynomial if it is invariant under this involution. For example, the anticommutator $A_1A_2 + A_2A_1$ is a selfadjoint polynomial, but the monomial A_1A_2 is not.

First we need to make sure that a selfadjoint polynomial of boson field operators is really a selfadjoint operator.

Lemma 4.5. *Let p be a selfadjoint polynomial in k variables. Then $p[F(\tilde{v}_1), \dots, F(\tilde{v}_k)]$ is essentially selfadjoint on $\mathcal{D}$ for any choice of $\tilde{v}_1, \dots, \tilde{v}_k \in \ell^2(\mathfrak{L})$.*

Proof. Take an orthonormal basis $\tilde{w}_1, \dots, \tilde{w}_l$ in the linear space spanned by $\tilde{v}_1, \dots, \tilde{v}_k$. The polynomial p is then identical to a new selfadjoint polynomial p_2 in the l variables $F(\tilde{w}_1), \dots, F(\tilde{w}_l)$. Denote $\mathcal{D}(\tilde{w}_i)$ the linear span of the vectors $F(\tilde{w}_i)^m\tilde{\Omega}$, $m \in \mathbb{N}$, then $\mathcal{D}$ can be decomposed as $\mathcal{D} = \left[\otimes_{i=1}^{l} \mathcal{D}(\tilde{w}_i)\right] \otimes \mathcal{D}'$ with p_2 acting on the first part only. Essential selfadjointness of p_2, and hence p, follows from Theorem VIII.33 of [21]. □

Theorem 4.6. *For all $k \in \mathbb{N}$, $v_1, \dots, v_k \in \ell^1_3(\mathfrak{L})$, and p a selfadjoint polynomial in k variables, we have*

$$\operatorname*{s-lim}_{J\to\infty} p\big[F_J(v_1), \dots, F_J(v_k)\big] = p\big[F(\tilde{v}_1), \dots, F(\tilde{v}_k)\big]$$

where the limit is taken over any sequence of $J \in \frac{1}{2}\mathbb{N}_0$ tending to ∞. More precisely, for all $n \in \mathbb{N}$, $\psi_n \in \mathcal{D}_n$, and $J \in \frac{1}{2}\mathbb{N}_0$ such that $2J > n + k$, we have

$$\begin{aligned}&\Big\|\Big[\prod_{j=1}^{k} F(\tilde{v}_j) - \prod_{j=1}^{k} F_J(v_j)\Big]\psi_n\Big\| \\ &\quad\le \frac{1}{(2J)^{1/2}}\Big[\prod_{j=1}^{k} |v_j|_1\Big]\Big[\frac{(n+k)!}{n!}\Big]^{1/2}\sum_{i=1}^{k} 2^{k+i}(n+k-i)^{1/2}\|\psi_n\| \qquad (11)\end{aligned}$$

Proof. By the previous lemma and Theorem VIII.25 of [21], it is again sufficient to prove eq. (11). We write the difference of products as a telescopic sum:

$$\begin{aligned}\prod_{j=1}^{k} F(\tilde{v}_j) - \prod_{j=1}^{k} F_J(v_j) = &\big[F(\tilde{v}_1) - F_J(v_1)\big]F(\tilde{v}_2)\dots F(\tilde{v}_k) \\ &+ F_J(v_1)\big[F(\tilde{v}_2) - F_J(v_2)\big]F(\tilde{v}_3)\dots F(\tilde{v}_k) \\ &+ \dots \\ &+ F_J(v_1)\dots F_J(v_{k-1})\big[F(\tilde{v}_k) - F_J(v_k)\big]\end{aligned}$$

We estimate each term separately, first using repeatedly Lemma 4.3, then using Lemma 4.4, and finally using repeatedly Lemma 4.2:

$$\begin{aligned}
&\Big\| F_J(v_1)\dots F_J(v_{i-1})\big[F(\tilde v_i) - F_J(v_i)\big]F(\tilde v_{i+1})\dots F(\tilde v_k)\psi_n\Big\| \\
&\le \Big[\prod_{j=1}^{i-1} 4|v_j|_1(n+k-j+1)^{1/2}\Big]\Big\|\big[F(\tilde v_i) - F_J(v_i)\big]F(\tilde v_{i+1})\dots F(\tilde v_k)\psi_n\Big\| \\
&\le \Big[\prod_{j=1}^{i-1} 4|v_j|_1(n+k-j+1)^{1/2}\Big]4|v_i|_1\frac{(n+k-i)}{(2J)^{1/2}}\Big\|F(\tilde v_{i+1})\dots F(\tilde v_k)\psi_n\Big\| \\
&\le \Big[\prod_{j=1}^{i-1} 4|v_j|_1(n+k-j+1)^{1/2}\Big]4|v_i|_1\frac{(n+k-i)}{(2J)^{1/2}} \\
&\quad\times \Big[\prod_{j=i+1}^{k} 2|\tilde v_j|_2(n+k-j+1)^{1/2}\Big]\|\psi_n\| \\
&\le \Big[\prod_{j=1}^{k} |v_j|_1\Big]4^i 2^{k-i}\frac{(n+k-i)^{1/2}}{(2J)^{1/2}}\Big[\prod_{j=1}^{k}(n+k-j+1)^{1/2}\Big]\|\psi_n\| \\
&= 2^{k+i}\Big[\prod_{j=1}^{k}|v_j|_1\Big]\Big[\frac{(n+k)!}{n!}\frac{(n+k-i)}{2J}\Big]^{1/2}\|\psi_n\| \quad \square
\end{aligned}$$

Now we return to the main subject of this paper, namely proving central limit theorems for the coherent state ω.

We get convergence of all moments as a trivial application of the Cauchy-Schwarz inequality and Theorem 4.6, more specifically eq. (11) for the case $n = 0$.

Corollary 4.7 (Moments). *For all $k \in \mathbb{N}$ and $v_1, \dots, v_k \in \ell^1_3(\mathfrak{L})$, we have*

$$\lim_{J\to\infty} \omega\Big(\prod_{j=1}^{k} F_J(v_j)\Big) = \tilde\omega\Big(\prod_{j=1}^{k} F_J(\tilde v_j)\Big)$$

where the limit is taken over any sequence of $J \in \frac{1}{2}\mathbb{N}_0$. More precisely, for all $J \in \frac{1}{2}\mathbb{N}_0$

$$\Big|\omega\Big(\prod_{j=1}^{k} F_J(v_j)\Big) - \tilde\omega\Big(\prod_{j=1}^{k} F_J(\tilde v_j)\Big)\Big| \le \frac{(k!)^{1/2}}{(2J)^{1/2}}\Big[\prod_{j=1}^{k}|v_j|_1\Big]\sum_{i=1}^{k} 2^{k+i}(k-i)^{1/2}$$

It is clear that Theorem 4.6 contains much more information about the convergence of the fluctuation operators than the convergence of all moments that is derived from it, or the convergence of characteristic functions that is given in Theorem 2.3. In the usual setting of quantum central limit theorems [7, 8, 10] a space on which all fluctuation operators as well as the limiting boson field operator act simultaneaously, does not exist and therefore the question of strong (or any other) operator convergence does not make sense in these situations. It is an interesting question, however, to ask whether the existing central limit theorems in the usual setting for sums of random variables can be strengthened. Therefore, we end this section with several reformulations of Theorem 4.6 to obtain statements that do make sense although they have not been proved in general. In Section 5.2 we will give a first example of such a stronger convergence result.

Corollary 4.8 (Spectral measure). *For any $k \in \mathbb{N}$, $v_1, \ldots, v_k \in \ell^1_3(\mathfrak{L})$, and p a selfajoint polynomial in k variables, the spectral measure of $p[F_J(v_1), \ldots, F_J(v_k)]$ in the coherent state ω converges to the spectral measure of $p[F(\tilde{v}_1), \ldots, F(\tilde{v}_k)]$ in the Fock state $\tilde{\omega}$, where convergence is as functionals on $\mathcal{C}_b(\mathbb{R})$, the bounded continuous functions on $\mathbb{R}$.*

Proof. It follows from Theorem 4.6 and Theorem VIII.20 of [21] that for any $f \in \mathcal{C}_b(\mathbb{R})$, and any $\psi \in \mathcal{F}$

$$\lim_{J\to\infty} f\Big(p\big[F_J(v_1), \ldots, F_J(v_k)\big]\Big)\psi = f\Big(p\big[F(\tilde{v}_1), \ldots, F(\tilde{v}_k)\big]\Big)\psi \quad \square$$

The following reformulation is probably the most interesting. It generalizes Theorem 2.3 by proving convergence of the characteristic function of arbitrary selfadjoint polynomials of fluctuation operators instead of only allowing linear combinations of them.

Corollary 4.9 (Characteristic function of polynomials). *For any $k \in \mathbb{N}$, $v_1, \ldots, v_k \in \ell^1_3(\mathfrak{L})$, and p a selfadjoint polynomial in k variables, we have*

$$\lim_{J\to\infty} \omega\Big(e^{ip[F_J(v_1),\ldots,F_J(v_k)]}\Big) = \tilde{\omega}\Big(e^{ip[F(\tilde{v}_1),\ldots,F(\tilde{v}_k)]}\Big)$$

where the limit is taken over any sequence $J \in \frac{1}{2}\mathbb{N}_0$ tending to ∞.

Proof. By Theorem 4.6 and Trotter's theorem (Theorem VIII.21 of [21]), we have

$$\operatorname*{s-lim}_{J\to\infty} e^{ip[F_J(v_1),\ldots,F_J(v_k)]} = e^{ip[F(\tilde{v}_1),\ldots,F(\tilde{v}_k)]}$$

The result follows from the Cauchy-Schwarz inequality in $\mathcal{F}$. $\square$

Remark 4.10. In the previous corollary we could further generalize and consider l selfadjoint polynomials p_j in k_j variables. It follows by the same argument that

$$\lim_{J\to\infty}\omega\Big(\prod_{j=1}^{l} e^{ip_j[F_J(v_{j,1}),\ldots,F_J(v_{j,k_j})]}\Big) = \tilde{\omega}\Big(\prod_{j=1}^{l} e^{ip_j[F(\tilde{v}_{j,1}),\ldots,F(\tilde{v}_{j,k_j})]}\Big)$$

In addition all products of polynomials and characteristic functions of polynomials (e.g., $p_1 \times e^{ip_2}$) can be considered as well.

Remark 4.11. We have always taken $v \in \ell_3^1(\mathfrak{L})$. This is convenient because then the finite J fluctuation operators $F_J(v)$ are bounded operators with

$$\|F_J(v)\| \le 2^{3/2}|v|_1 J^{1/2}$$

We can ask about more general examples where $F_J(v)$ is only densely defined, and one such generalization is the following. Consider in expression (8) the last term

$$\Big(\frac{2}{J}\Big)^{1/2}\sum_{x\in\mathfrak{L}} \tilde{v}_x^3 a_x^* a_x$$

On $\mathcal{D}_n$ this operator is bounded by $2^{1/2}J^{-1/2}n\sup_{x\in\mathfrak{L}}|\tilde{v}_x^3|$. Hence all of our results remain unchanged, except for notationally more complicated error bounds, if we consider $v \in \tilde{\ell}_3(\mathfrak{L})$ instead of $v \in \ell_3^1(\mathfrak{L})$, where $\tilde{\ell}_3(\mathfrak{L})$ is defined as the set of $\mathbb{R}^3$-valued sequences on $\mathfrak{L}$ such that $\tilde{v} = \{(\tilde{v}_x^1, \tilde{v}_x^2)\}_{x\in\mathfrak{L}} \in \ell^1(\mathfrak{L})$ and $\tilde{v}^3 = \{\tilde{v}_x^3\}_{x\in\mathfrak{L}} \in \ell_1^\infty(\mathbb{Z})$.

Remark 4.12. In all of the previous results, we can replace the state $\Omega_J = \tilde{\Omega}$ by a perturbed state $\tilde{\Omega}_P$, as long as the perturbed state is still in the set $\mathcal{D}$ of analytic vectors for $F(\tilde{v})$.

5. Applications

5.1. Bosonization for quantum spin Hamiltonians

The main application of Theorem 4.6 is for the large-spin asymptotics of quantum spin Hamiltonians and their corresponding dynamics. Consider for instance an interaction between spins in a finite volume Λ of the type

$$H_{J,\Lambda} = -\sum_{x,y\in\Lambda}\sum_{i,j=1}^{3} h_{ij}(x,y)S_x^i S_y^i - J\sum_{x\in\Lambda}\sum_{i=1}^{3} g_i(x)S_x^i \tag{12}$$

where the interaction functions h_{ij} and g_i satisfy all necessary conditions for selfadjointness of $H_{J,\Lambda}$, and in addition are bounded and of short enough range, i.e.,

$$\sup_{x\in\mathfrak{L}}\Big(\sum_{y\in\mathfrak{L}}\big|h_{ij}(x,y)\big|\Big) < \infty$$

$$\sup_{x\in\mathfrak{L}}|g_i(x)| < \infty$$

More general Hamiltonians can easily be treated along the same lines, but this example already includes the various interesting Heisenberg-type models. The minus sign in front of the Hamiltonian is there for convenience, we do not make any explicit assumptions on the sign of the h_{ij} or g_i.

Bosonization refers to the idea that the low-energy excitations of a quantum spin system can be effectively described by a boson approximation to (12), describing non-interacting bosons hopping on the lattice $\mathfrak{L}$. To make this into a mathematically precise statement, the Hamiltonian has to be scaled by J^{-1} and a large-spin limit has to be taken. Again there exist various ways of taking this limit. One way consists in proving that the free energy at low temperatures converges to the free energy of a boson model, see, e.g. [4–6, 22], and this has been used mainly to study the isotropic Heisenberg ferromagnet. Another approach was introduced in [18] in the study of 1-dimensional anisotropic Heisenberg ferromagnet, and consists of proving strong operator convergence of the Hamiltonian like we did for the fluctuation observables in Theorem 4.6. From there, convergence of eigenvalues and eigenvectors is derived.

In both approaches it is necessary that there is a unique ground state to be used as a reference state, and this ground state should be a coherent product state. Both the isotropic and anisotropic Heisenberg model have a rotational symmetry, and hence uniqueness of the ground state can only be achieved by adding an external field which breaks this symmetry. For the isotropic model, there is full rotational symmetry, and the final results do not depend on which ground state is selected. For the anisotropic model, there exist non translation invariant ground states describing interfaces, and the approximating boson model is different for different ground states, see [18] for details.

In this section we want to demonstrate another use of Theorem 4.6, namely in studying the dynamics of (ground) states under perturbations. The main assumption is that there exists a ground state ω of (12) which is a product state of coherent states, like we used throughout this paper. For the present application it is not necessary that it is the unique ground state.

The first step is to write the Hamiltonian (12) in the new natural basis for the state ω:

$$H_{J,\Lambda} = -\sum_{x,y\in\Lambda}\sum_{i,j=1}^{3} \tilde{h}_{ij}(x,y)\tilde{S}_x^i\tilde{S}_y^j - J\sum_{x\in\Lambda}\sum_{i=1}^{3}\tilde{g}_i(x)\tilde{S}_x^i$$

where the $\tilde{h}_{ij}$ and $\tilde{g}_i$ are easily obtained using the basis transformation on page 499. Instead of taking $i, j = 1, 2, 3$ we can also take $i, j = -, +, 3$. The coherent state in the new basis is just the 'all up' state, and this is an eigenstate of $H_{J,\Lambda}$ if and only if the only non-zero coefficients are as follows:

$$\begin{aligned} H_{J,\Lambda} = &-\sum_{x,y\in\Lambda}\Big[\tilde{h}_{-+}(x,y)\tilde{S}_x^-\tilde{S}_y^+ + \tilde{h}_{+-}(x,y)\tilde{S}_x^+\tilde{S}_y^- + \tilde{h}_{33}(x,y)\tilde{S}_x^3\tilde{S}_y^3\Big] \\ &- J\sum_{x\in\Lambda}\tilde{g}_3(x)\tilde{S}_x^3 - \sum_{x,y\in\Lambda}\Big[\tilde{h}_{13}(x,y)\tilde{S}_x^1\tilde{S}_y^3 + \tilde{h}_{31}(x,y)\tilde{S}_x^3\tilde{S}_y^1 \\ &+ \tilde{h}_{23}(x,y)\tilde{S}_x^2\tilde{S}_y^3 + \tilde{h}_{32}(x,y)\tilde{S}_x^3\tilde{S}_y^2\Big] \end{aligned}$$

For the terms on the first line, no additional conditions except for those of selfadjointness and summability are needed, the terms on the second line however can only be allowed if for all $x \in \Lambda$

$$\sum_{y\in\Lambda} \tilde{h}_{i3}(x,y) = \sum_{y\in\Lambda} \tilde{h}_{3i}(x,y) = 0 \ \text{ for } \ i = 1, 2$$

Suppose the reference state ω is a ground state of $H_{J,\Lambda}$. Then, its energy is given by

$$-\sum_{x,y\in\Lambda} \tilde{h}_{33}(x,y) - \sum_{x\in\Lambda} \tilde{g}_3(x)$$

Hence we can renormalize the Hamiltonian so that the ground state energy is 0, and obtain

$$\begin{aligned} H_{J,\Lambda} = &-\sum_{x,y\in\Lambda} \Big[\tilde{h}_{-+}(x,y)\tilde{S}^-_x\tilde{S}^+_y + \tilde{h}_{+-}(x,y)\tilde{S}^+_x\tilde{S}^-_y + \tilde{h}_{33}(x,y)\big(\tilde{S}^3_x\tilde{S}^3_y - J^2\big)\Big] \\ &- J\sum_{x\in\Lambda} \tilde{g}_3(x)\big(\tilde{S}^3_x - J\big) \\ &- \sum_{x,y\in\Lambda} \Big[\tilde{h}_{13}(x,y)\tilde{S}^1_x\big(\tilde{S}^3_y - J\big) + \tilde{h}_{31}(x,y)\big(\tilde{S}^3_x - J\big)\tilde{S}^1_y \\ &\quad + \tilde{h}_{23}(x,y)\tilde{S}^2_x\big(\tilde{S}^3_y - J\big) + \tilde{h}_{32}(x,y)\big(\tilde{S}^3_x - J\big)\tilde{S}^2_y\Big] \end{aligned}$$

With the Hamiltonian correctly renormalized, we can also write down the GNS Hamiltonian H_J acting on the GNS Hilbert space $\mathcal{H}_J$, it is the same expression with all sums extended from Λ to the entire lattice $\mathfrak{L}$.

It is also clear that after scaling with J^{-1}, this Hamiltonian is of the form

$$\begin{aligned} \tfrac{1}{J}H_J = &-\sum_{x,y\in\mathfrak{L}} p^{(2)}_{xy}\big(F_J(f^1_x), F_J(f^2_x), F_J(f^3_x)\big) \\ &+\frac{1}{J}\sum_{x,y\in\mathfrak{L}} \tilde{h}_{33}(x,y)\big(J^2 - \tilde{S}^3_x\tilde{S}^3_y\big) + \sum_{x\in\Lambda} \tilde{g}_3(x)\big(J - \tilde{S}^3_x\big) \end{aligned}$$

where $p^{(2)}_{xy}$ are selfadjoint polynomials in 2 of the 3 listed variables, and f^i_x are the rotated basis vectors defined on page 499.

Using Theorem 4.6, we get immediately that

$$\begin{aligned} \operatorname*{s-lim}_{J\to\infty} &\sum_{x,y\in\mathfrak{L}} p^{(2)}_{xy}\big(F_J(f^1_x), F_J(f^2_x), F_J(f^3_x)\big) \\ &= \sum_{x,y\in\Lambda} \Big[\tilde{h}_{-+}(x,y)a^*_x a_y + \tilde{h}_{+-}(x,y)a_x a^*_y\Big] \end{aligned}$$

The remaining terms can be evaluated by writing them on Fock space:

$$\sum_{x\in\Lambda} \tilde{g}_3(x)\big(J - \tilde{S}^3_x\big) = \sum_{x\in\Lambda} \tilde{g}_3(x) n_x$$

and

$$\begin{aligned}\frac{1}{J}\sum_{x,y\in\mathfrak{L}}\tilde{h}_{33}(x,y)\big(J^2-\tilde{S}_x^3\tilde{S}_y^3\big) &= \sum_{x,y\in\mathfrak{L}}\tilde{h}_{33}(x,y)\Big[J-\tfrac{1}{J}(J-n_x)(J-n_y)\Big]\\ &= \sum_{x,y\in\mathfrak{L}}\tilde{h}_{33}(x,y)(n_x+n_y-\tfrac{1}{J}n_xn_y)\\ &= \sum_{x\in\mathfrak{L}}E(x)n_x-\sum_{x,y\in\mathfrak{L}}\tilde{h}_{33}(x,y)\frac{n_xn_y}{J}\end{aligned}$$

where

$$E(x)=\sum_{y\in\mathfrak{L}}\big[\tilde{h}_{33}(x,y)+\tilde{h}_{33}(y,x)\big]$$

Using $\sum_x n_x = n\mathbb{1}$ on $\mathcal{D}_n$, it is seen that the second term above converges strongly to 0.

Hence we conclude:

$$\begin{aligned}\operatorname*{s-lim}_{J\to\infty}\frac{1}{J}H_J=\tilde{H}:=&\sum_{x\in\mathfrak{L}}\big[E(x)+\tilde{g}_3(x)\big]a_x^*a_x\\ &-\sum_{x,y\in\Lambda}\big[\tilde{h}_{-+}(x,y)a_x^*a_y+\tilde{h}_{+-}(x,y)a_xa_y^*\big]\end{aligned}\tag{13}$$

and this is a rigorous way to express bosonization in a quantum spin model which is of the type (12) and has a coherent ground state. As the coefficients $\tilde{g}_3(x)$, $\tilde{h}_{-+}(x,y)$, etc., depend on the rotation of the basis, it is clear from this derivation that the boson Hamiltonian in general will be different for different ground states. Notice also that it is the second quantization of an operator H on $\ell^2(\mathfrak{L})$ defined by:

$$(Hv)_x=\big[E(x)+\tilde{g}_3(x)\big]v_x-\sum_{y\in\mathfrak{L}}\big[\tilde{h}_{-+}(x,y)+\tilde{h}_{+-}(y,x)\big]v_y\tag{14}$$

In [18], eq. (13) is used to show that the eigenvalues and eigenvectors of the boson Hamiltonian give the large-spin asymptotics of the eigenvalues and eigenvectors of the spin Hamiltonian (for the special case of the anisotropic ferromagnetic Heisenberg chain). This is a very practical result because the spectrum of the spin Hamiltonian is generally very hard, if not impossible, to compute for large systems, while the spectrum of the boson Hamiltonian, which is quadratic, can be computed very easily, also for large systems (certainly numerically).

Theorem 4.6 can also be used to obtain a rigorous estimate for another kind of computation, namely for the dynamics of states, typically eigenstates of H_J, under perturbations of the Hamiltonian. Such a problem is typically studied in a weak coupling limit, where the strength of the perturbation vanishes as the microscopic time tends to infinity, see [19] for a recent example studying the dynamics of interfaces.

Clearly, if we perturb the spin Hamiltonian by any selfadjoint polynomial in the fluctuation operators, we can study the influence of this perturbation using the bosonization approximation, i.e., taking the large-spin limit. More precisely, let p be a selfadjoint polynomial in k variables and let $v_1, \ldots, v_k \in \ell^1_3(\mathfrak{L})$. Denote

$$P_J = p\big[F_J(v_1), \ldots, F_J(v_k)\big] \qquad \tilde{P} = p\big[F(\tilde{v}_1), \ldots, F(\tilde{v}_k)\big]$$

Then

$$\operatorname*{s\text{-}lim}_{J\to\infty} \tfrac{1}{J} H_J + P_J = \tilde{H} + \tilde{P}$$

and again by Trotter's theorem (Theorem VIII.21 of [21]), for all $\psi \in \mathcal{F}$,

$$\lim_{J\to\infty} e^{it(\frac{1}{J}H_J + P_J)}\psi = e^{it(\tilde{H}+P)}\psi$$

While the spin expression on the l.h.s. can again be very difficult to compute, the boson expression on the r.h.s. can be solved for relevant cases.

Consider for instance the case where the Hamiltonian is perturbed by a fluctuation of the spin in a certain direction, i.e., $P_J = F_J(v)$ for some $v \in \ell^1_3(\mathfrak{L})$. Such a perturbation was also considered in [9] in the study of spatial fluctuations of quantum spin systems and their relation with linear response theory. Using a Dyson expansion and the CCR algebraic structure we find

$$e^{it[\tilde{H}+F(\tilde{v})]}e^{-it\tilde{H}} = e^{-\frac{i}{2}\int_0^t ds\sigma(\tilde{v},\tilde{v}^s)}e^{iF(\tilde{v}^t)}$$

where

$$\tilde{v}^t = \int_0^t ds e^{isH}\tilde{v}$$

and H is given by (14). Hence the boson approximation to the time evolution of the ground state under the perturbed evolution is given by

$$e^{it(\tilde{H}+P)}\tilde{\Omega} = e^{it[\tilde{H}+F(\tilde{v})]}e^{-it\tilde{H}}\tilde{\Omega} = e^{-\frac{i}{2}\int_0^t ds\sigma(\tilde{v},\tilde{v}^s)}e^{iF(\tilde{v}^t)}\tilde{\Omega}$$

Using techniques such as in the previous section and such as in Lemma A.2 below, explicit estimates which vanish for large values of J can be obtained to compare this state with the true evolved state

$$e^{it(\frac{1}{J}H_J + P_J)}\Omega_J$$

5.2. *Strong central limit theorem for the large N limit of N spin-$\frac{1}{2}$ particles*

In this section, we show that Corollary 4.9 gives a fully noncommutative example of Theorem 1 of [14]. In this example, we consider a limit where the number of spin variables, N, tends to infinity, while the magnitude of each spin remains constant equal to 1/2. We change the notation accordingly.

A quantum probability space consists of a C^* or von Neumann algebra $\mathfrak{A}$, and a state ω on $\mathfrak{A}$. Then $(\mathfrak{A}^{\otimes N}, \omega_N = \omega^{\otimes N})$ denotes N independent copies of $(\mathfrak{A}, \omega)$, and if $A \in \mathfrak{A}$, then

$$A_i = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

with A in the i^{th} position, denotes the i^{th} copy of A.

The fluctuation operator of $A \in \mathfrak{A}$ in this context is defined as

$$F_N(A) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \big[A_i - \omega(A)\big]$$

and we can ask the same convergence questions as we did before. In fact, this simple setup was the first in which the connection between quantum fluctuation operators and representations of the CCR was worked out in detail [7]. In that paper, the result analogous to Theorem 2.3 was first proved.

The question whether stronger central limit theorems holds in this setup was first raised in [13], and in [14] the following special case was proved:

Theorem (Kuperberg [14]). *Let* $(\mathfrak{A}, \tau)$ *be a quantum probability space with* τ *a* tracial *state. For all* $k \in \mathbb{N}$, $A_1, \ldots, A_k$ *selfadjoint elements of* $\mathfrak{A}$, *and* $p \in \mathbb{C}\langle F_N(A_1), \ldots, F_N(A_k)\rangle$ *a selfadjoint polynomial in k variables, we have*

$$\lim_{N\to\infty} \tau_N\Big(e^{ip[F_N(A_1),\ldots,F_N(A_k)]}\Big) = \mathbb{E}\Big(e^{ip[X(A_1),\ldots,X(A_k)]}\Big)$$

where $X(A_1), \ldots, X(A_k)$ *are* classical *Gaussian random variables with covariance matrix*

$$\mathbb{E}\big(X(A_i)X(A_j)\big) = \tau(A_i A_j)$$

The fact that a tracial state leads to classical fluctuations in the limit is to be expected, it follows directly from the cyclicity of the trace. As far as we know, no generalization of this theorem to non-tracial states exists.

Consider the simple case in which $\mathfrak{A} = \mathbb{M}_2(\mathbb{C})$, the complex (2×2)-matrices. $\mathfrak{A}$ carries the irreducible spin-$\frac{1}{2}$ representation of $SU(2)$ given by $\frac{1}{2}$ times the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The standard basis of $\mathbb{C}^2$ is diagonal for σ^3, we denote $|1\rangle = \binom{1}{0}$ and $|-1\rangle = \binom{0}{1}$. For the state we take $\omega = \omega_{\theta,\varphi}$, the expectation in the spin-$\frac{1}{2}$ coherent state

$$|(\theta, \varphi)\rangle = e^{\frac{1}{4}\theta(\sigma^- e^{i\varphi} - \sigma^+ e^{-i\varphi})}|1\rangle$$

where $\sigma^{\pm} = \sigma^1 \pm i\sigma^2$.

We are interested in the selfadjoint operators

$$v \cdot \sigma = v^1\sigma^1 + v^2\sigma^2 + v^3\sigma^3, \quad v \in \mathbb{R}^3$$

and their fluctuation operators $F_N(v) := F_N(v \cdot \sigma)$.

But a collection of N independent spin-$\frac{1}{2}$ degrees of freedom can also be regarded as one spin-$\frac{N}{2}$ degree of freedom, i.e., define

$$S_N^j = \sum_{i=1}^{N} \tfrac{1}{2}\sigma_i^j$$

then these operators satisfy the $SU(2)$ commutators as well, and the eigenvalues of each S^j obviously are $-\frac{N}{2}, -\frac{N}{2}+1, \dots, \frac{N}{2}$. Moreover we have in the coherent state ω:

$$\begin{aligned}\omega_N\big(e^{iv\cdot S}\big) &= \omega_N\big(e^{\frac{i}{2}\sum_i v\cdot\sigma_i}\big) = \prod_{i=1}^{N} \omega\big(e^{\frac{i}{2}v\cdot\sigma}\big) \\ &= \Big\{\cos\big(\tfrac{1}{2}|v|\big) + i\frac{v\cdot u}{|v|}\cos\big(\tfrac{1}{2}|v|\big)\Big\}^N\end{aligned}$$

where $u = (\theta, \varphi)$. This is the correct generating function for the spin-$\frac{N}{2}$ coherent state, cfr. eq. (2). Hence the present setup is exactly equivalent with the setup in the previous sections if we take the lattice $\mathfrak{L}$ consisting of a single point, and check that the normalization of the fluctuation operators is consistent with eq. (3):

$$\sqrt{\frac{2}{\frac{1}{2}N}}\big[v\cdot S_N - \omega_N(v\cdot S_N)\big] = \frac{2}{\sqrt{N}}\sum_{i=1}^{N}\big[\tfrac{1}{2}v\cdot\sigma_i - \tfrac{1}{2}\omega(v\cdot\sigma_i)\big] = F_N(v)$$

It follows that Corollary 4.9 remains valid in the present setup, and we get immediately the follwoing theorem.

Theorem 5.1. *Consider the quantum probability space* $(\mathbb{M}_2(\mathbb{C}), \omega)$ *where* ω *is the coherent state defined by the unit vector* $u = (\theta, \varphi) \in \mathbb{R}^3$. *For all* $k \in \mathbb{N}$, $v_1, \dots, v_k \in \mathbb{R}^3$, *and* $p \in \mathbb{C}\langle F_N(v_1), \dots, F_N(v_k)\rangle$ *a selfadjoint polynomial in k variables, we have*

$$\lim_{N\to\infty} \omega_N\Big(e^{ip[F_N(v_1),\dots,F_N(v_k)]}\Big) = \tilde{\omega}\Big(e^{ip[F(\tilde{v}_1),\dots,F(\tilde{v}_k)]}\Big)$$

where $\tilde{\omega}$ *is the Fock state on* $\mathsf{CCR}(\mathbb{R}^2, \sigma)$, σ *is the standard symplectic form, and* $F(\tilde{v})$ *are the boson field operators for* $\tilde{v} \in \mathbb{R}^2$ *which is obtained from* $v \in \mathbb{R}^3$ *by projecting onto the tangent plane to the unit sphere at u.*

Note that the limiting object in the above CLT, the Fock state on $\mathsf{CCR}(\mathbb{R}^2, \sigma)$, is essentially the quantum harmonic oscillator. It can be regarded as a non-commuting pair ("position" and "momentum") of Gaussian random variables.

Appendix A. Proofs for Section 2

Although the results in Section 2 are very similar to the standard central limit theorems of [7, 10, 8], their derivation requires different methods. In this appendix, we highlight these differences and leave the details of the computations to the reader.

Proof of Proposition 2.2. The characteristic function of $F_J(v)$ can be computed using eq. (2):

$$\omega\big(e^{itF_J(v)}\big) = \prod_x \bigg\{ \frac{1}{2}\Big(1 + \frac{v_x \cdot u_x}{|v_x|}\Big) \exp\Big[\frac{it|v_x|}{(2J)^{1/2}}\Big(1 - \frac{v_x \cdot u_x}{|v_x|}\Big)\Big] + \frac{1}{2}\Big(1 - \frac{v_x \cdot u_x}{|v_x|}\Big) \exp\Big[\frac{it|v_x|}{(2J)^{1/2}}\Big(1 + \frac{v_x \cdot u_x}{|v_x|}\Big)\Big]\bigg\}^{2J} \tag{15}$$

From the formula

$$\omega\big(e^{itF_J(v)}\big) = \exp\Big\{\sum_{k=1}^{\infty} \frac{(it)^k}{k!} \omega_T\big(\underbrace{F_J(v), \ldots, F_J(v)}_{k}\big)\Big\}$$

it follows that the k-point truncated correlation function is given by

$$\begin{aligned}\omega_T\big(\underbrace{F_J(v), \ldots, F_J(v)}_{k}\big) &= \Big(\frac{1}{i}\frac{d}{dt}\Big)^k \ln \omega\big(e^{itF_J(v)}\big)\Big|_{t=0} \\ &= 2J \sum_x \Big(\frac{1}{i}\frac{d}{dt}\Big)^k \ln f_x(t)\Big|_{t=0}\end{aligned}$$

where $f_x(t)$ is the part between $\{\ldots\}$ in (15). We find for $k \geq 2$:

$$\begin{aligned}&\omega_T\big(\underbrace{F_J(v), \ldots, F_J(v)}_{k}\big) \\ &\quad = \frac{1}{2(2J)^{k/2-1}} \sum_x |v_x|^k \Big(1 - \frac{(v_x \cdot u_x)^2}{|v_x|^2}\Big) \\ &\quad\quad \times \Big\{\Big(1 - \frac{v_x \cdot u_x}{|v_x|}\Big)^{k-1} + (-1)^k \Big(1 + \frac{v_x \cdot u_x}{|v_x|}\Big)^{k-1}\Big\}\end{aligned}$$

For $k = 2$ this reduces to eq. (5), i.e., $\omega_T(F_J(v), F_J(v)) = |\tilde{v}|_2^2$, while for $k > 2$, we get an upper bound

$$\begin{aligned}\Big|\omega_T\big(\underbrace{F_J(v), \ldots, F_J(v)}_{k}\big)\Big| &\leq \frac{1}{(2J)^{k/2-1}} \sum_x |v_x|^k \Big(1 + \frac{|v_x \cdot u_x|}{|v_x|}\Big)^{k-1} \leq \frac{2^{k/2}|v|_k^k}{J^{k/2-1}} \\ &\leq \frac{2^{k/2}|v|_2^k}{J^{k/2-1}}\end{aligned}$$

which is sufficient to prove Proposition 2.2. □

Proof of Proposition 2.1. The proof follows from the previous proof by considering the centred observable

$$M_J(v) = \frac{1}{J} \sum_x \big(v_x \cdot S_x - J(v_x \cdot u_x)\big) = \frac{1}{(2J)^{1/2}} F_J(v)$$

□

The first step in the proof of Theorem 2.3 is the following lemma.

Lemma A.1 (A BCH-formula). *Let $v_1, v_2 \in \ell^2_3(\mathfrak{L})$, then, for $J \in \frac{1}{2}\mathbb{N}_0$,*

$$\left\| e^{iF_J(v_1)} e^{iF_J(v_2)} - e^{iF_J(v_1+v_2)} e^{-\frac{1}{2}[F_J(v_1), F_J(v_2)]} \right\| \leq \frac{1}{3J^{1/2}} |v_1|_2 |v_2|_2 \big(|v_1|_2 + |v_2|_2\big)$$

where $\|\cdot\|$ is the operator norm in $\mathcal{B}(\mathcal{H}_J)$.

In the usual setup of quantum central limit theorems, the BCH-formula is proved by an explicit calculation exploiting the central limit scaling and summation of quantum random variables. Here we prove the following general lemma, which implies the BCH-formula:

Lemma A.2. *Let $\mathfrak{A}$ be a $\mathcal{C}^*$-algebra, $A, B \in \mathfrak{A}$ selfadjoint, and $C \in \mathfrak{A}$ arbitrary, then*

$$\left\| [e^{iA}, C] \right\| \leq \| [A, C] \| \tag{16}$$

$$\left\| e^{iA} e^{iB} - e^{i(A+B)} e^{-\frac{1}{2}[A,B]} \right\| \leq \frac{1}{3}\Big(\left\| [A, [A, B]] \right\| + \left\| [B, [B, A]] \right\| \Big) \tag{17}$$

Proof. Eq. (16) is well-known and follows from the identity

$$[e^{isA}, C] = i \int_0^s dt\, e^{i(s-t)A} [A, C] e^{itA} \tag{18}$$

To prove (17), we start from

$$\left\| e^{iA} e^{iB} - e^{i(A+B)} e^{-\frac{1}{2}[A,B]} \right\| \leq \left\| 1 - e^{-iB} e^{-iA} e^{i(A+B)} e^{-\frac{1}{2}[A,B]} \right\|$$

Let

$$F(t) = 1 - e^{-iB} e^{-itA} e^{i(tA+B)} e^{-\frac{t}{2}[A,B]}$$

Using (18), we can compute the derivative $F'(t)$:

$$F'(t) = \int_0^1 ds \int_0^s du\; e^{-iB} e^{-itA} e^{i(s-u)(tA+B)} \big[e^{i(1-s+u)(tA+B)}, [A, B] \big] e^{-\frac{t}{2}[A,B]}$$

Since

$$F(1) = F(0) + \int_0^1 dt\; F'(t)$$

we find, using (16),

$$\begin{aligned} \|F(1)\| &\leq \int_0^1 dt \int_0^1 ds \int_0^s du\; |1 - s + u| \left\| [tA + B, [A, B]] \right\| \\ &\leq \frac{1}{3}\Big(\left\| [A, [A, B]] \right\| + \left\| [B, [B, A]] \right\| \Big) \end{aligned}$$

□

Proof of Theorem 2.3. The result follows from Proposition 2.2 and Lemma A.1 by a standard induction argument which also uses Proposition 2.1. It can be found in [7, 10]. The main difference is that we keep track of the error estimates in Proposition 2.2 and Lemma A.1 to also obtain an error estimate in Theorem 2.3. □

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