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The Large-Spin Asymptotics of the Ferromagnetic XXZ Chain*

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Abstract. We present new results and give a concise review of recent previous results on the asymptotics for large spin of the low-lying spectrum of the ferromagnetic XXZ Heisenberg chain with kink boundary conditions. Our main interest is to gain detailed information on the interface ground states of this model and the low-lying excitations above them. The new and most detailed results are obtained using a rigorous version of bosonization, which can be interpreted as a quantum central limit theorem.

Keywords: XXZ chain, Heisenberg ferromagnet, large-spin limit, bosonization

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1. Introduction

In recent years the XXZ model has become a popular model to study properties of interface states in quantum lattice models. As an interpolation between the Ising model and the isotropic (XXX) Heisenberg ferromagnet, the ferromagnetic XXZ model has the interesting features of both. By considering the Ising model and the XXX model as limiting cases of the XXZ model, intuition about these two limits can be used to better understand the XXZ model. In this paper we are interested in the large-spin asymptotics of the low-lying excitation spectrum of the XXZ chain, in particular the excitations above the kink (or interface) ground states of the model. In a nutshell, our main result is that the spin-wave approximation, in the sense of Dyson [6], becomes exact in the limit

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of infinitely large spin. The technical statements are given in Section 2. First, we introduce the model and the main notations and give a quick summary of the relevant previous results.

For $J = 1/2, 1, 3/2, \ldots$, the spin-J XXZ Hamiltonian on an interval $\Lambda = [a,b] \subset \mathbf{Z}$, with kink boundary conditions, is given by

$$H_{J,\Lambda} = \sum_{x=a}^{b-1} H^J_{x,x+1}, \tag{1.1}$$
$$H^J_{x,x+1} = J^2 - \frac{1}{\Delta}(S^1_x S^1_{x+1} + S^2_x S^2_{x+1}) - S^3_x S^3_{x+1} + J\sqrt{1-\Delta^{-2}}(S^3_x - S^3_{x+1})$$

with S^i_x the spin-J matrices acting on site x:

$$[S^i_x, S^j_y] = i\delta_{x,y}\varepsilon_{ijk}S^k_x, \qquad S_x \cdot S_x = (S^1_x)^2 + (S^2_x)^2 + (S^3_x)^2 = J(J+1).$$

We will also use the spin raising and lowering operators: S^+_x and S^-_x, $S^\pm_x = S^1_x \pm iS^2_x$.

We begin with a brief overview of the main results obtained for the Hamiltonians $H_{J,\Lambda}$. The spin 1/2 model, $J = 1/2$, is Bethe Ansatz solvable and possesses a quantum group symmetry [19]. Consequently, there are a number of results specific to the spin 1/2 case (e.g., see [3,11,18]). Since the main focus in this paper is on large-J behavior, we will not discuss these specific results here.

The Hamiltonian (1.1) is symmetric under global rotations around the 3-axis generated by $S^3_{\mathrm{tot},\Lambda} = \sum_{x\in\Lambda} S^3_x$, which represents the third component of the total magnetization. Hence, $H_{J,\Lambda}$ is block diagonal, and it is known that in each sector corresponding to a given eigenvalue of $S^3_{\mathrm{tot},\Lambda}$ there is exactly one ground state, i.e., in each sector 0 is a simple eigenvalue [12]:

$$H_{J,\Lambda}\Phi^{(M)}_\Lambda = 0, \quad S^3_{\mathrm{tot},\Lambda}\Phi^{(M)}_\Lambda = M\Phi^{(M)}_\Lambda,$$

where $M = -|\Lambda|J, -|\Lambda|J+1, \ldots, |\Lambda|J$. The (unnormalized) eigenvector $\Phi^{(M)}_\Lambda$ is given by

$$\Phi^{(M)}_\Lambda = \sum_{\substack{\{m_x\} \\ \sum_x m_x = M}} \prod_{x\in\Lambda} \binom{2J}{J-m_x}^{1/2} q^{x(J-m_x)} |\{m_x\}\rangle,$$

where we introduced the parameter q, $0 < q < 1$, by the equation $2\Delta = q + q^{-1}$. These ground states have a magnetization profile that shows an interface, or *kink*, with a location depending on the value of M. For a short review on the properties of these ground states see [17].

In many instances it is important to consider the thermodynamic limit, i.e, the limit of infinitely long chains. To this end, we consider a strictly increasing sequence of numbers $a_n \in \mathbf{Z}^+$ and $\lim_n a_n = \infty$, and a sequence of volumes $\Lambda_n = [-a_n + 1, a_n]$. The set of eigenvalues of $S^3_{\mathrm{tot},\Lambda_n}$ is then

$$\mathcal{M}_n = \{-2a_n J, -2a_n J + 1, \ldots, 2a_n J\}$$

and since $2J$ is an integer, we have $\mathcal{M}_n \subset \mathcal{M}_m$ for $n < m$. Hence we can fix $M \in \mathbf{Z}$, take n_0 large enough such that $M \in \mathcal{M}_n$ for all $n \geq n_0$, and consider a sequence of states

$$\omega^{(M)}_{\Lambda_n} = \frac{\langle \Phi^{(M)}_{\Lambda_n}, \,\cdot\, \Phi^{(M)}_{\Lambda_n} \rangle}{\langle \Phi^{(M)}_{\Lambda_n}, \Phi^{(M)}_{\Lambda_n} \rangle}.$$

It is shown in [10,13] that in the limit $n \to \infty$, any such sequence converges in norm to a unique state $\omega^{(M)}$ on the quasi-local algebra of observables $\mathfrak{A}$ which is the norm completion of the algebra of local observables given by

$$\mathfrak{A}_{\mathrm{loc}} = \bigcup_{\Lambda \subset \mathrm{Z}} \bigotimes_{x \in \Lambda} \mathbf{M}_{2J+1}(\mathbf{C})$$

and each $\omega^{(M)}$ is a ground state for the derivation δ_J defined by

$$\delta_J(X) = \lim_{\Lambda \nearrow \mathrm{Z}} i[H_{J,\Lambda}, X],$$

i.e.,

$$\omega^{(M)}\big(X^* \delta(X)\big) \geq 0 \quad \text{for } X \in \mathfrak{A}_{\mathrm{loc}}.$$

All these infinite volume kink states have the same GNS Hilbert space, namely the incomplete tensor product Hilbert space

$$\mathcal{H}_J = \overline{\bigcup_{\Lambda \subset \mathrm{Z}} \Big(\Big[\bigotimes_{x \in \Lambda} \mathbf{C}^{2J+1} \Big] \otimes \Big[\bigotimes_{y \in \mathrm{Z} \setminus \Lambda} \Omega_y \Big] \Big)}$$

where

$$\Omega_y = \begin{cases} |-J\rangle, & \text{if } y \leq 0, \\ |J\rangle, & \text{if } y > 0. \end{cases}$$

Also denote $\Omega = \otimes_{y \in \mathrm{Z}} \Omega_y \in \mathcal{H}_J$, and the GNS Hamiltonian on $\mathcal{H}_J$ will be denoted H_J.

It was proved in [13,21] that, for all $J = 1/2, 1, 3/2, \ldots$, these Hamiltonians have a gap above the ground state eigenvalue, which is 0. Let us denote the gap by $\gamma_{J,M}$. In the case of $J = 1/2$, the exact value of the gap was previously known to be $1 - \Delta^{-1}$, for all $\Delta \geq 1$. In [13] a very explicit conjecture is made about the value of the gap in the limit $J \to \infty$. For all finite J, it is a periodic function of M, with period $2J$. The conjecture in [13] is as follows:

Conjecture 1.1. *For all $\mu \in \mathbf{R}$, the limit $\lim_{J\to\infty}(1/J)\gamma_{J,\mu J} = \tilde{\gamma}(r)$ exists and is given by the smallest positive eigenvalue of the Jacobi operator $\tilde{h}^{(r)}$, defined below in* (2.1)*, where r is the solution of the equation*

$$\mu = \sum_{x=-\infty}^{+\infty} \tanh(\eta(x-r)) \tag{1.2}$$

with η determined by $\Delta = \cosh \eta$.

A partial result towards this conjecture was proved in [4]. Namely, there it is shown that there are constants $c_1 > 0$, and $c_2 > 0$, independent of M and J, such that

$$c_1 J \leq \gamma_{J,M} \leq c_2 J \ .$$

In this paper we prove that the value of the gap claimed in the conjecture is asymptotically correct. The results in this paper by themselves, however, do not amount to a proof of the conjecture as stated. Roughly speaking, we obtain the result in the "grand-canonical ensemble", and with the aid of a ground state selection mechanism that localizes the kink. The conjecture is stated in the "canonical ensemble", i.e., for fixed $\mu = M/J$, in which the kink is automatically localized at a fixed location. As is often the case the distinction between canonical and grand-canonical results seems merely technical, but proving mathematical equivalence of both formulations is often highly non-trivial. In fact, equivalence of ensembles in the usual sense does not hold in the present situation. To prove the conjecture as stated above, some additional work has to be done. We will report on this further work in a future publication [16].

The conjecture of [13] was based on results form perturbation theory and numerical calculations on small systems presented in [21], as well as on a heuristic calculation leading to a Boson model.

The idea is to apply a rigorous version of Dyson's spin wave formalism to the XXZ chain. Mathematically speaking, the task is to control the quadratic approximation, described by a quasi-free system of Bosons, and show that this approximation becomes exact in the limit $J \to \infty$.

Several authors have attempted to do this for the XXX model, with interesting results [2, 22]. In these works, the authors considered the XXX model in an external magnetic field, and it was necessary to let the strength of the field diverge as $J \to \infty$. Such a field selects a particular ground state (out of the infinite number of them), and creates a gap in the spectrum. This allows one to proceed, but it limits the mathematical applicability of the spin-wave formalism. In the case of the XXZ-model, the situation is somewhat better. First, the XXZ chain by itself (i.e., without external magnetic field) already has a non-vanishing spectral gap. Second, although the infinite XXZ chain also has an infinite number of ground states — with the degeneracy now corresponding to the arbitrary position of the kink — any field at just one site with a non-vanishing component in the XY-plane, a so-called pinning field, will select a unique ground state, for finite J [5]. Moreover, the magnitude of this field, as we will show, can be taken of smaller order in J. These features of the XXZ model will allow us to prove asymptotic properties of the model itself.

2. Main results

2.1. The limiting Boson model

Our main result will be that the spectrum of the XXZ chain, in the limit of infinite spin, can be understood as the spectrum of a non-interacting system of Bosons on the chain, with one-particle Hamiltonian, $\tilde{h}^{(r)}$, defined on $\ell^2(\mathbf{Z})$ by

$$(\tilde{h}^{(r)}v)_x = \varepsilon_x v_x - \Delta^{-1}(v_{x-1} + v_{x+1}), \tag{2.1}$$

where

$$\varepsilon_x = \frac{2\cosh(\eta(x-r))^2}{\cosh(\eta(x-1-r))\cosh(\eta(x+1-r))}$$

with $\eta = -\ln q$ or, equivalently, $\Delta = \cosh\eta$, and $r \in \mathbf{R}$ is the position of the kink in the reference ground state (see (1.2)). $\tilde{h}^{(r)}$ has the form of the discrete Laplacian (kinetic energy) plus a diagonal term given by ε_x, which is an exponentially localized potential well centered around the interface.

We list some properties of $\tilde{h}^{(r)}$, some of which are easily proved, while other more detailed properties about its spectrum have at this point only been verified numerically. We will discuss these in more detail elsewhere.

(i) $\tilde{h}^{(r)}$is a positive operator;

(ii) $\tilde{h}^{(r)}$has an eigenvalue 0 with eigenvector $v_0 \in \ell^2(\mathbf{Z})$, up to normalization defined by $v_{0,x} = 1/\cosh(\eta(x-r))$;

(iii) the bottom of the continuous spectrum of $\tilde{h}^{(r)}$ is given by $2(1-\Delta^{-1})$;

(iv) the first excited state of $\tilde{h}^{(r)}$corresponds to an isolated eigenvalue $\tilde{\gamma}^{(r)}$ below the continuum;

(v) the (Δ^{-1}, r)-plane is divided in a region where $\tilde{\gamma}^{(r)}$is the only eigenvalue below the continuum, and a region where there is another isolated eigenvalue between $\tilde{\gamma}^{(r)}$and the bottom of the continuum.

Let $\tilde{h}^{(r)}(x,y)$ be the bi-infinite Jacobi matrix expressing (2.1) in the standard Kronecker-delta basis of $\ell^2(\mathbf{Z})$, i.e.,

$$\tilde{h}^{(r)}(x,y) = \varepsilon_x \delta_{x,y} - \frac{1}{\Delta}(\delta_{x-1,y} + \delta_{x+1,y}).$$

The Boson Hamiltonian is then given by second quantization of $\tilde{h}^{(r)}$:

$$\tilde{H}^{(r)} = \sum_{x,y\in \mathrm{Z}} \tilde{h}^{(r)}(x,y) a_x^* a_y,$$

where a_x^* and a_y are the creation and annihilation operators for a Boson at site x and y, respectively. They act on the Bosonic Fock space with one-particle space $\ell^2(\mathbf{Z})$, $\mathcal{F}$, and satisfy the canonical commutation relations (CCR)

$$a_y a_x^* - a_x^* a_y = \delta_{x,y}\mathbf{1}, \quad a_y a_x - a_x a_y = a_y^* a_x^* - a_x^* a_y^* = 0, \quad x, y \in \mathbf{Z}.$$

Let $\tilde{\Omega} \in \mathcal{F}$ denote the vacuum vector which, up to a scalar factor is uniquely characterized by the property $a_x\tilde{\Omega} = 0$, for all $x \in \mathbf{Z}$.

We will often use the following standard orthonormal basis in $\mathcal{F}$. Introduce

$$\vec{n} = \{n_x \in \mathbf{N}\}_{x\in \mathrm{Z}}, \qquad \mathcal{N} = \Big\{\vec{n} \,\Big|\, \sum_x n_x < \infty\Big\}.$$

Then, the set $\{\varphi_{\vec{n}} \mid \vec{n} \in \mathcal{N}\}$, where

$$\varphi_{\vec{n}} = \prod_{x\in \mathrm{Z}} \frac{1}{(n_x!)^{1/2}} (a_x^*)^{n_x} \tilde{\Omega}, \tag{2.2}$$

is an orthonormal basis of $\mathcal{F}$.

The GNS Hilbert spaces of the spin chains, $\mathcal{H}_J$, $J \geq 1/2, 1, 3/2, \ldots$, can be identified with a nested sequence of subspaces of $\mathcal{F}$, defined for each J, as the linear span of all vectors $\varphi_{\vec{n}}$, with $n_x \leq 2J$. We will use this identification throughout the paper, and we will use the same symbol $\varphi_{\vec{n}}$ to denote a vector in the spin Hilbert space $\mathcal{H}_J$ and the boson space $\mathcal{F}$. More precisely, we will use the projections $P_{n,x}$ on $\mathcal{F}$ which project onto the first $2n$ boson states at site x, i.e., on the states $\varphi_{\vec{n}}$ with $0 \leq n_x \leq 2n$, and denote $P_n = \prod_x P_{n,x}$, and find

$$\mathcal{H}_J = P_J \mathcal{F}.$$

More details on the Boson model are given in Section 3.3.

2.2. Ground state selection by the external field

To prove full convergence of the low-energy spectrum, we need to add an external field. A physical external field would be of the form

$$\sum_x \vec{h}_{J,x} \cdot \vec{S}_x. \tag{2.3}$$

For our purposes, however, the field is a perturbation and our results will generally be more interesting if we can prove them with smaller perturbations.

Ideally, a vanishingly small field localized at one site should suffice to select a reference ground state. It turns out that we cannot quite do this in the present setup. We shall use a perturbation of the form

$$\sup_x \big(\vec{h}_{J,x} \cdot \vec{S}_x\big), \tag{2.4}$$

which is still significantly smaller than a uniform field. The meaning of this operator is clear if we express states in a basis which is diagonal for each $\vec{h}_{J,x} \cdot \vec{S}_x$. It is important to remark that we can take

$$|\vec{h}_{J,x}| \equiv h_J = h(J \ln J)^{2/3},$$

such that after scaling with J^{-1} the external field vanishes, in contrast with the fields employed in previous works on the XXX model [2, 22]. In fact some results can be obtained with $h_J \equiv 0$, or $h_J = h \ln J$.

Mathematically, the field (2.4) does slightly more than a field localized at one site can achieve. It not only pins the interface, but also puts some control on the local fluctuations around the selected ground state, which cannot be controlled otherwise. The fact that we can let the field vanish as J increases, and that we do not need a global field like (2.3), are signs that these fluctuations are smaller in the XXZ model than in the isotropic model.

There is another mechanism of selecting a ground state, namely by restricting the full Hilbert space to a subspace of states with fixed total magnetization in the 3-direction, since it is known that in each such sector there is exactly one ground state, see Section 1. The ground states that are pinned by an external field are like grand-canonical averages of the canonical ground states with fixed magnetization. In the limit $J \to \infty$ the canonical description can be obtained from the grand canonical one by a result analogous to the result of [21, Sections 5.11, 5.12] about equivalence of ensembles in the 2-dimensional, spin-1/2 XXZ model. Note however that there is *no* equivalence of ensembles in our situation, in the sense that correlation functions are typically different. But the difference between a canonical state and a grand-canonical state with the right average magnetization can be expressed completely in terms of the fluctuations of the total 3-magnetization in the grand canonical state which are non-zero even in the limit $J \to \infty$, while they are identically zero in the canonical state. The results about the canonical description require significant additional work and will be discussed elsewhere [16].

Finally we mention that because of the pinning field (2.4), our results give only a partial proof of Conjecture 1.1. A full proof requires in addition that the lowest excited state can be obtained from the ground state by acting on it with 1 spin wave operator, or more generally by a finite number of operators independent of J. This is a problem that should be handled at the level of the spin system, rather than in the spin wave formalism. Some results in that direction have recently been obtained in [18].

An advantage of the grand-canonical description is that it clearly exhibits how the Boson limit arises as the first quantum correction to the classical limit.

2.3. Statement of the main results

For A a self-adjoint operator and $a, b \in \mathbf{R}$, denote by $P_{(a,b)}(A)$ the spectral projection of A onto (a,b). For A acting on Fock space, denote

- the spectrum of A in $\mathcal{F}$ by $\sigma(A)$;
- the spectrum of $P_J A P_J$ in $\mathcal{H}_J$ by $\sigma_J(A)$;
- the spectrum of $P_{n_J} A P_{n_J}$ in $\mathcal{H}_{n_J}$ by $\sigma_{n_J}(A)$, where $\mathcal{H}_{n_J}$ is shorthand for $P_{n_J}\mathcal{F}$.

Also denote s-lim the strong, or strong resolvent, operator limit for bounded, resp. unbounded operators acting on $\mathcal{F}$.

In the GNS space $\mathcal{H}_J$ it is convenient to define S^3_{tot} in the renormalized sense: $S^3_{\text{tot}} = \sum_{x \in \mathrm{Z}} \left[S^3_x - \operatorname{sgn}(x - 1/2)\right]$, and denote

$$\mu = \sum_{x \in \mathrm{Z}} \left[\tanh(\eta(x-r)) - \operatorname{sgn}\left(x - \frac{1}{2}\right)\right]$$

instead of (1.2).

Theorem 2.1. *We have*

$$\operatorname*{s-lim}_{J\to\infty} \frac{1}{J} H_J = \tilde{H}^{(r)}, \qquad \operatorname*{s-lim}_{J\to\infty} \frac{1}{J} S^3_{\text{tot}} = \mu \mathbf{1}.$$

The proof of this result is given in Section 5.2.

Corollary 2.1.

(i) *If $\lambda \in \sigma(\tilde{H}^{(r)})$, there exists $\lambda_J \in \sigma_J((1/J)H_J)$ such that*

$$\lim_{J\to\infty} \lambda_J = \lambda.$$

(ii) *If $a, b \in \mathbf{R}$, and $a, b \notin \sigma_{pp}(\tilde{H}^{(r)})$, then*

$$\operatorname*{s-lim}_{J\to\infty} P_{(a,b)}\left(\frac{1}{J} H_J\right) = P_{(a,b)}(\tilde{H}^{(r)}).$$

(iii) *If $a, b \in \mathbf{R}$, and $\mu \in (a,b)$, then*

$$\operatorname*{s-lim}_{J\to\infty} P_{(a,b)}\left(\frac{1}{J} S^3_{\text{tot}}\right) = \mathbf{1}.$$

This corollary is a direct consequence of Proposition 5.1 and [20, Theorem VIII.24].

In addition we can obtain spectral concentration of $(1/J)H_J$ around discrete eigenvalues of $\tilde{H}^{(r)}$ (also proved in Section 5.2).

Theorem 2.2. *For every isolated eigenvalue E of $\tilde{H}^{(r)}$, there exists an interval*

$$I_J = (E - \varepsilon_J, E + \varepsilon_J) \quad \text{with} \quad \lim_{J\to\infty} \varepsilon_J \frac{J}{\ln J} = 0,$$

such that for any interval I around E s.t. $I \cap \sigma(\tilde{H}^{(r)}) = \{E\}$:

$$\operatorname*{s-lim}_{J\to\infty} P_{I\setminus I_J}\Big(\frac{1}{J}H_J\Big) = 0, \qquad \operatorname*{s-lim}_{J\to\infty} P_{I_J}\Big(\frac{1}{J}H_J\Big) = P_{\{E\}}\big(\tilde{H}^{(r)}\big).$$

Applying the same reasoning to S^3_{tot}, we find that the interval (a,b) in item (iii) of Corollary 5.2 can be chosen as

$$(a,b) = (\mu - \varepsilon_J, \mu + \varepsilon_J)$$

with again $\lim_J \varepsilon_J J(\ln J)^{-1} = 0$.

To prove full convergence of the spectrum, we have to add the external field (2.4) to H_J, or, through the identification $\mathcal{H}_J = P_J\mathcal{F}$, add:

$$h_J \sup_x (N_x) \tag{2.5}$$

with $h_J > 0$, and $N_x = a^*_x a_x$. Let us assume we add this field to $(1/J)H^J$, so h_J already contains the factor J^{-1}.

Take $0 < n_J < J$, we get that on $(\mathcal{H}_{n_J})^\perp \cap \mathcal{H}_J$

$$\frac{1}{J}H_J + h_J \sup_x N_x \geq h_J \sup_x N_x \geq h_J n_J \mathbf{1}.$$

Clearly, by choosing h_J and n_J such that

$$\lim_{J\to\infty} h_J n_J = \infty$$

statements about the spectrum on $\mathcal{H}_J$ reduce to statements about the spectrum on $\mathcal{H}_{n_J}$. Or, if one chooses to make statements about the spectrum below a certain value E, it is sufficient to choose h_J such that $\lim_J h_J n_J > E + \varepsilon$.

Theorem 2.3. *Let $n_J = [(J(\ln J)^{-1})^{1/3}]$. If $\lambda \notin \sigma(\tilde{H}^{(r)})$, then $\lambda \notin \sigma_{n_J}((1/J) \times H_J)$ for J large enough.*

This result is proved in Section 5.3.

Hence we get convergence of the spectrum of $(1/J)H_J + h_J \sup_x N_x$ to the spectrum of $\tilde{H}^{(r)}$ if

$$\lim_{J\to\infty} h_J = 0, \qquad \lim_{J\to\infty} h_J \Big(\frac{J}{\ln J}\Big)^{1/3} = \infty.$$

3. Derivation of the Boson limit

3.1. The classical limit

The Boson limit can be considered as the first quantum correction to the classical limit. Therefore, we first discuss the classical limit.

It is well known that for any quantum spin system, after rescaling each spin matrix by J^{-1} and taking the large spin limit $J \to \infty$, one obtains the corresponding classical spin system [14]. For the XXZ chain in a finite volume this is defined by the Hamiltonian

$$H_\Lambda^{\rm cl}\big(\{\sigma_x\}_{x\in\Lambda}\big) = \sum_{x=a}^{b-1} 1 - \frac{1}{\Delta}\big(\sigma_x^1\sigma_{x+1}^1 + \sigma_x^2\sigma_{x+1}^2\big) - \sigma_x^3\sigma_{x+1}^3 + \sqrt{1-\Delta^{-2}}\big(\sigma_x^3 - \sigma_{x+1}^3\big), \tag{3.1}$$

where σ_x is a unit vector in $\mathbf{R}^3$. Minimizing this function with respect to $\{\sigma_x\}_x$ yields zero-energy configurations that are planar waves [21], i.e., in spherical coordinates we find configurations $\sigma_x^{(r)}(\varphi) = (\theta_x^{(r)}, \varphi)$, $\varphi \in [-\pi, \pi]$ (the same at all sites), and $\theta_x^{(r)} = 2\arctan(q^{x-r})$, where $r \in \mathbf{R}$ determines the value of the total 3-magnetization.

Defining again $\eta = -\ln q$ or $\Delta = \cosh(\eta)$, we have

$$\cos\theta_x^{(r)} = \frac{1-q^{2(x-r)}}{1+q^{2(x-r)}} = \tanh\big(\eta(x-r)\big), \tag{3.2}$$

$$\sin\theta_x^{(r)} = \frac{2q^{x-r}}{1+q^{2(x-r)}} = \frac{1}{\cosh\big(\eta(x-r)\big)}, \tag{3.3}$$

such that the zero-energy solutions clearly describe kinks centered around r.

For the classical model, to look at the low-energy behavior amounts to making a quadratic Taylor approximation to (3.1). At each site, the angle coordinates are replaced by new coordinates

$$q_x = \theta_x - \theta_x^{(r)}, \qquad p_x = \sin\theta_x^{(r)}(\varphi_x - \varphi),$$

and the resulting harmonic oscillator Hamiltonian is

$$\begin{aligned}\tilde{H}_\Lambda^{\rm cl} = \frac{1}{2}\sum_{x=a}^{b-1}\Big(\varepsilon_x^+(q_x^2+p_x^2) - \frac{1}{\Delta}\big(q_xq_{x+1}+p_xp_{x+1}\big)\Big)\\ + \frac{1}{2}\sum_{x=a+1}^{b}\Big(\varepsilon_x^-(q_x^2+p_x^2) - \frac{1}{\Delta}\big(q_xq_{x-1}+p_xp_{x-1}\big)\Big),\end{aligned} \tag{3.4}$$

where $\varepsilon_x^\pm$ are given by

$$\varepsilon_x^\pm = \frac{\sin\theta_{x\pm1}^{(r)}}{\Delta\sin\theta_x^{(r)}} = \frac{\cosh(\eta(x-r))}{\Delta\cosh(\eta(x\pm1-r))}.$$

This can be derived using the identities of Lemma 5.1 below.

At sites other than the boundary sites we have a single-site potential

$$\varepsilon_x = \varepsilon_x^+ + \varepsilon_x^- = \frac{2\cosh(\eta(x-r))^2}{\cosh(\eta(x-1-r))\cosh(\eta(x+1-r))}$$

which is an exponentially localized well centered around the interface.

3.2. Grand canonical states

For (θ_x, φ_x) a general unit vector on the sphere at site x, we can define the coherent spin state in $(\mathbf{C}^{2J+1})_x$ (see [1, 14]):

$$\begin{aligned}|(\theta_x, \varphi_x)\rangle &= \exp\left\{\frac{1}{2}\theta_x\big(S_x^- e^{i\varphi_x} - S_x^+ e^{i\varphi_x}\big)\right\}|J\rangle \\ &= \sum_{m_x=-J}^{J} \binom{2J}{J-m_x}^{1/2} \left(\cos\frac{1}{2}\theta_x\right)^{J+m_x} \left(\sin\frac{1}{2}\theta_x\right)^{J-m_x} \\ &\quad \times \exp\{i(J-m_x)\varphi_x\}|m_x\rangle.\end{aligned}$$

This is particularly interesting if we choose the unit vectors at each site to be the classical zero-energy configurations. In a finite volume Λ, it is easy to see that

$$\big|\sigma_\Lambda^{(r)}(\varphi)\big\rangle \equiv \bigotimes_{x\in\Lambda} e^{-iJ\varphi}\big|(\theta_x^{(r)}, \varphi)\big\rangle = \frac{1}{\|\Psi_\Lambda^{(z)}\|}\Psi_\Lambda^{(z)},$$

where $\Psi_\Lambda^{(z)}$is the generating vector for the ground state vectors $\Phi_\Lambda^{(M)}$, i.e., the grand canonical ground state:

$$\Psi_\Lambda^{(z)} = \sum_{M=-|\Lambda|J}^{|\Lambda|J} z^M \Phi_\Lambda^{(M)}$$

evaluated at $z = q^r \exp\{-i\varphi\} = \exp\{-\eta r\}\exp\{-i\varphi\}$.

The fact that a classical ground state yields an exact quantum ground state through the coherent state representation, is because $H_{J,\Lambda}$ is a normal Hamiltonian in the sense of [14], and the classical and quantum ground state energies are (exactly) related by the scaling factor J^2.

Since these states are product states, their thermodynamic limit is easily obtained. In the GNS Hilbert space $\mathcal{H}_J$, define the embedding of $|\sigma_\Lambda^{(r)}(\varphi)\rangle$ as

$$\big|\bar{\sigma}_\Lambda^{(r)}(\varphi)\big\rangle \equiv \big|\sigma_\Lambda^{(r)}(\varphi)\big\rangle \otimes \big[\bigotimes_{x\in \mathbf{Z}\setminus\Lambda} \Omega_x\big].$$

Lemma 3.1. *For a sequence of intervals* $\Lambda_n = [-a_n + 1, a_n]$ *tending to* $\mathbf{Z}$, *we have for* $m > n$

$$\big\| \, |\bar\sigma^{(r)}_{\Lambda_m}(\varphi)\rangle - |\bar\sigma^{(r)}_{\Lambda_n}(\varphi)\rangle \big\| \leq 2Jq^{2a_n}\, \frac{1-q^{2(a_m-a_n)}}{1-q^2}\big(q^{2r}+q^{2-2r}\big).$$

Proof.

$$\begin{aligned}
&\big\| \, |\bar\sigma^{(r)}_{\Lambda_m}(\varphi)\rangle - |\bar\sigma^{(r)}_{\Lambda_n}(\varphi)\rangle \big\|^2 \\
&\quad = \Big\| \bigotimes_{x\in\Lambda_m\setminus\Lambda_n} e^{-iJ\varphi}|(\theta^{(r)}_x,\varphi)\rangle - \bigotimes_{x\in\Lambda_m\setminus\Lambda_n} \Omega_x \Big\|^2 \\
&\quad = 2 - \prod_{x=-a_m+1}^{-a_n} e^{-iJ\varphi}\langle -J \mid (\theta^{(r)}_x,\varphi)\rangle \prod_{x=a_n+1}^{a_m} e^{-iJ\varphi}\langle J \mid (\theta^{(r)}_x,\varphi)\rangle \\
&\qquad - \prod_{x=-a_m+1}^{-a_n} e^{iJ\varphi}\langle (\theta^{(r)}_x,\varphi) \mid -J\rangle \prod_{x=a_n+1}^{a_m} e^{iJ\varphi}\langle (\theta^{(r)}_x,\varphi) \mid J\rangle \\
&\quad = 2 - 2\prod_{x=a_n}^{a_m-1} \frac{1}{(1+q^{2(x+r)})^J} \prod_{x=a_n+1}^{a_m} \frac{1}{(1+q^{2(x-r)})^J} \\
&\quad \leq 2\Big(1 - \exp\Big(-J\sum_{x=a_n}^{a_m-1} q^{2(x+r)} - J\sum_{x=a_n+1}^{a_m} q^{2(x-r)}\Big)\Big) \\
&\quad \leq 2J\Big(\sum_{x=a_n}^{a_m-1} q^{2(x+r)} + \sum_{x=a_n+1}^{a_m} q^{2(x-r)}\Big) \\
&\quad = 2Jq^{2a_n}\,\frac{1-q^{2(a_m-a_n)}}{1-q^2}\big(q^{2r}+q^{2-2r}\big),
\end{aligned}$$

where we used the inequalities (for $u \geq 0$) $1+u \leq \exp u$ and $1-\exp\{-u\} \leq u$. □

It follows that the sequence $|\bar\sigma^{(r)}_{\Lambda_n}(\varphi)\rangle$ has a limit $|\sigma^{(r)}(\varphi)\rangle$ in $\mathcal{H}_J$ that we can formally write as

$$\begin{aligned}
\big|\sigma^{(r)}(\varphi)\big\rangle \equiv{}& \Big[\bigotimes_{x\leq 0} \exp\Big\{-\frac{1}{2}(\pi-\theta^{(r)}_x)(S^-_x e^{i\varphi} - S^+_x e^{i\varphi})\Big\}\Omega_x\Big] \\
&\otimes \Big[\bigotimes_{x>0} \exp\Big\{\frac{1}{2}\theta^{(r)}_x(S^-_x e^{i\varphi} - S^+_x e^{i\varphi})\Big\}\Omega_x\Big].
\end{aligned}$$

In general, if we write $z = |z|e^{-i\varphi}$, then

$$\Psi^{(z)}_\Lambda = \exp\{-i\varphi S^3_{\mathrm{tot},\Lambda}\}\Psi^{(|z|)}_\Lambda, \tag{3.5}$$

so it will be sufficient to restrict our detailed analysis to the grand canonical states $\Psi^{(e^{-\eta r})}_\Lambda$. The expectation in this state will be denoted $\omega^{(r)}_\Lambda$:

$$\omega_\Lambda^{(r)} = \frac{\langle \Psi_\Lambda^{(e^{-\eta r})}, \, \cdot \, \Psi_\Lambda^{(e^{-\eta r})} \rangle}{\left\| \Psi_\Lambda^{(e^{-\eta r})} \right\|^2}$$

and its thermodynamic limit $\omega^{(r)}$.

In this case, the coherent states are rotations of the 'top' state $|J\rangle$ through an angle $\theta_x^{(r)}$ around the 2-axis, $|(\theta_x^{(r)}, 0)\rangle = \exp\{-i\theta_x^{(r)} S_x^2\}|J\rangle$. Introduce the notation $u_x^{(r)} = (\theta_x^{(r)}, 0)$, or in Cartesian coordinates $u_x^{(r)} = (\sin\theta_x^{(r)}, 0, \cos\theta_x^{(r)})$.

In the remainder, we will always keep r fixed and do not make explicit the dependence on r of various quantities. Notice that by periodicity it is sufficient to take $r \in [0, 1)$.

Denote by $\{e_x^1, e_x^2, e_x^3\}$ the standard basis in $\mathbf{R}^3$ (the same at every site), and

$$\begin{aligned} f_x^1 &= \cos\theta_x^{(r)} \; e_x^1 - \sin\theta_x^{(r)} e_x^3, \\ f_x^2 &= e_x^2, \\ f_x^3 &= u_x^{(r)} = \sin\theta_x^{(r)} \; e_x^1 + \cos\theta_x^{(r)} e_x^3. \end{aligned}$$

Conversely

$$\begin{aligned} e_x^1 &= \cos\theta_x^{(r)} \; f_x^1 + \sin\theta_x^{(r)} f_x^3, \\ e_x^2 &= f_x^2, \\ e_x^3 &= -\sin\theta_x^{(r)} \; f_x^1 + \cos\theta_x^{(r)} f_x^3, \end{aligned}$$

i.e., $\{f_x^1, f_x^2, f_x^3\}$ form an orthonormal frame for $\mathbf{R}^3$ and $\{f_x^1, f_x^2\}$ an orthonormal frame for the tangent plane $\mathbf{R}^2$ to the unit sphere at $u_x^{(r)}$.

For $v_x \in \mathbf{R}^3$, we denote by $\tilde{v}_x \in \mathbf{R}^2$ the projection of v_x onto the tangent plane at $u_x^{(r)}$ (shifted to the origin), i.e., $\tilde{v}_x = (\tilde{v}_x^1, \tilde{v}_x^2)$ and

$$\tilde{v}_x^1 = v_x \cdot f_x^1 = \cos\theta_x^{(r)} v_x^1 - \sin\theta_x^{(r)} v_x^3, \qquad \tilde{v}_x^2 = v_x \cdot e_x^2 = v_x^2.$$

Conversely, if $\tilde{v}_x \in \mathbf{R}^2$ we associate to it a vector $v_x \in \mathbf{R}^3$ by putting the component along the $u_x^{(r)}$-axis zero:

$$v_x^1 = \cos\theta_x^{(r)} \tilde{v}_x^1, \qquad v_x^2 = \tilde{v}_x^2, \qquad v_x^3 = -\sin\theta_x^{(r)} \tilde{v}_x^1.$$

Also denote

$$v_x \cdot S_x = v_x^1 S_x^1 + v_x^2 S_x^2 + v_x^3 S_x^3$$

and define rotated spin operators [1, Eq. (3.9)]

$$\tilde{S}_x^1 = \exp\{-i\theta_x^{(r)} S_x^2\} S_x^1 \exp\{i\theta_x^{(r)} S_x^2\} = f_x^1 \cdot S_x = \cos\theta_x^{(r)} S_x^1 - \sin\theta_x^{(r)} S_x^3, \quad (3.6)$$

$$\tilde{S}_x^2 = \exp\{-i\theta_x^{(r)} S_x^2\} S_x^2 \exp\{i\theta_x^{(r)} S_x^2\} = S_x^2, \quad (3.7)$$

$$\tilde{S}_x^3 = \exp\{-i\theta_x^{(r)} S_x^2\} S_x^3 \exp\{i\theta_x^{(r)} S_x^2\} = f_x^3 \cdot S_x = \sin\theta_x^{(r)} S_x^1 + \cos\theta_x^{(r)} S_x^3. \quad (3.8)$$

Hence we find that $|(\theta_x^{(r)},0)\rangle$ is the 'top' state for the rotated spin operators:

$$\tilde{S}_x^3\big|(\theta_x^{(r)},0)\big\rangle = \tilde{S}_x^3 \exp\{-i\theta_x^{(r)}S_x^2\}|J\rangle = \exp\{-i\theta_x^{(r)}S_x^2\}S_x^3|J\rangle = J\big|(\theta_x^{(r)},0)\big\rangle. \tag{3.9}$$

The rotated spin raising and lowering operators are

$$\tilde{S}_x^\pm = \tilde{S}_x^1 \pm i\tilde{S}_x^2 = -\sin\theta_x^{(r)}S_x^3 + \cos\theta_x^{(r)}S_x^1 \pm iS_x^2,$$

or

$$\begin{aligned}\tilde{S}_x^+ &= -\sin\theta_x^{(r)}S_x^3 + \cos^2\frac{\theta_x^{(r)}}{2}S_x^+ - \sin^2\frac{\theta_x^{(r)}}{2}S_x^-,\\ \tilde{S}_x^- &= -\sin\theta_x^{(r)}S_x^3 - \sin^2\frac{\theta_x^{(r)}}{2}S_x^+ + \cos^2\frac{\theta_x^{(r)}}{2}S_x^-.\end{aligned}$$

One of the main observations is that because of (3.9), it is much more convenient to introduce the spin wave formalism in the rotated spin basis than in the original one. Following [22], introduce

$$\vec{n} = \{n_x \in \mathbf{N}\}_{x\in \mathrm{Z}},$$

$$\mathcal{N} = \Big\{\vec{n} \mid \sum_x n_x < \infty\Big\}, \qquad \mathcal{N}_J = \Big\{\vec{n} \mid \forall x\colon n_x \le 2J, \sum_x n_x < \infty\Big\}, \tag{3.10}$$

$$\varphi_{\vec{n}} = \prod_{x\in\mathrm{Z}} \frac{1}{n_x!}\binom{2J}{n_x}^{-1/2}\big(\tilde{S}_x^-\big)^{n_x}\Omega^{(r)}. \tag{3.11}$$

The set $\{\varphi_{\vec{n}} \mid \vec{n}\in\mathcal{N}_J\}$ is an orthonormal basis for $\mathcal{H}_J$.

We conclude with a little lemma that complements (3.5).

Lemma 3.2. *For* $-\pi < \varphi < \pi$,

$$\begin{aligned}\exp\{i\varphi S^3_{\mathrm{tot},\Lambda}\}\big|\sigma_\Lambda^{(r)}\big\rangle &= \big(\cos(\varphi/2) + i\cos\theta_x^{(r)}\sin(\varphi/2)\big)^{2J}\\ &\quad\times\exp\Big\{-i\sum_{x\in\Lambda}\alpha_x(\varphi)\sin\theta_x^{(r)}\tilde{S}_x^-\Big\}\big|\sigma_\Lambda^{(r)}\big\rangle,\end{aligned}$$

where

$$\alpha_x(\varphi) = \frac{\sin(\varphi/2)}{\cos(\varphi/2) + i\cos\theta_x^{(r)}\sin(\varphi/2)}.$$

Proof. We write the disentanglement relation [1, Eq. (A4)]

$$\begin{aligned}\exp\{i\varphi S_x^3\} &= \exp\Big\{i\varphi\Big(-\frac{1}{2}\sin\theta_x^{(r)}(\tilde{S}_x^+ + \tilde{S}_x^-) + \cos\theta_x^{(r)}\tilde{S}_x^3\Big)\Big\}\\ &= \exp\{-iy_-\tilde{S}_x^-\}\exp\{(\ln y_3)\tilde{S}_x^3\}\exp\{iy_+\tilde{S}_x^+\},\end{aligned}$$

where

$$y_3 = \big(\cos(\varphi/2) + i\cos\theta_x^{(r)}\sin(\varphi/2)\big)^2,$$
$$y_+ = y_- = \frac{\sin\theta_x^{(r)}\sin(\varphi/2)}{\cos(\varphi/2) + i\cos\theta_x^{(r)}\sin(\varphi/2)} = \alpha_x(\varphi)\sin\theta_x^{(r)}$$

and recall that $|\sigma_\Lambda^{(r)}\rangle$ is the product state of 'top' states for the $\tilde{S}$-operators. □

It is now also clear how to choose the external field $\vec{h}_{J,x}$ in (2.3) such that $\vec{h}_{J,x}\cdot\vec{S}_x = -h_J\tilde{S}_x^3$, namely $\vec{h}_{J,x} = -h_J u_x^{(r)}$, $h_J > 0$.

3.3. The Boson chain

We consider immediately the infinite volume situation. Consider the Hilbert space of wave functions $\ell^2(\mathbf{Z})$ which we alternatively consider as the usual complex Hilbert space with inner product $\langle v, w\rangle = \sum_{x\in\mathrm{Z}}\overline{v}_x w_x$ or as a real linear space with symplectic form σ and complex structure $\mathcal{J}$, i.e.,

$$v \in \ell^2(\mathbf{Z}) = \big((v_x^1, v_x^2) \in \mathbf{R}^2\big)_{x\in\mathrm{Z}},$$
$$\sigma(v,w) = \sum_{x\in\mathrm{Z}} v_x^1 w_x^2 - v_x^2 w_x^1,$$
$$\mathcal{J}v = \big((-v_x^2, v_x^1) \in \mathbf{R}^2\big)_{x\in\mathrm{Z}}.$$

The CCR -algebra $\mathrm{CCR}(\ell^2(\mathbf{Z}),\sigma)$ is generated by unitaries $\{W(v) \mid v \in \ell^2(\mathbf{Z})\}$ which satisfy the commutation relations

$$W(v)W(w) = \exp\Big\{-\frac{i}{2}\sigma(v,w)\Big\}W(v+w).$$

The Fock state $\tilde{\omega}$ is the quasi-free state on $\mathrm{CCR}(\ell^2(\mathbf{Z}),\sigma)$ determined by

$$\tilde{\omega}\big(W(v)\big) = \exp\Big\{-\frac{1}{2}\langle v, v\rangle\Big\}.$$

Its GNS representation is the usual Fock representation on a Fock space $\mathcal{F}$ with a vacuum vector $\tilde{\Omega} = \otimes_{x\in\mathrm{Z}}|0\rangle_x$ and creation and annihilation operators $a_x^\sharp$ such that

$$W(v) = \exp\Big\{i\sum_{x\in\mathrm{Z}}(v_x a_x^* + \overline{v}_x a_x)\Big\}.$$

(We do not distinguish between $W(v)$ and its representative in the Fock representation).

In this Fock representation we can define a quasi-free Boson Hamiltonian by canonically quantizing the classical harmonic oscillator Hamiltonian (3.4). This means replacing the position and momentum variables by canonical pairs q_x, p_x, with commutation relations $[q_x, p_y] = i\delta_{x,y}$ and

$$a_x^* = \frac{q_x - ip_x}{\sqrt{2}}, \qquad a_x = \frac{q_x + ip_x}{\sqrt{2}}.$$

The result is

$$\tilde{H}_\Lambda^{(r)} = \sum_{x=a}^{b-1} \Big(\varepsilon_x^+ \Big(a_x^* a_x + \frac{1}{2} \Big) - \frac{1}{\Delta} a_x^* a_{x+1} \Big) + \sum_{x=a+1}^{b} \Big(\varepsilon_x^- \Big(a_x^* a_x + \frac{1}{2} \Big) - \frac{1}{\Delta} a_x^* a_{x-1} \Big).$$

The corresponding infinite volume derivation is denoted

$$\tilde{\delta}^{(r)}(\cdot) = \lim_{\Lambda \nearrow \mathrm{Z}} i\, [\tilde{H}_\Lambda^{(r)}, \cdot]$$

and the GNS Hamiltonian is denoted $\tilde{H}^{(r)}$,

$$\tilde{H}^{(r)} = \sum_{x \in \mathrm{Z}} \varepsilon_x a_x^* a_x - \Delta^{-1} a_x^* (a_{x-1} + a_{x+1}).$$

Denote $a^*(v) = \sum_x v_x a_x^*$. If v is local, i.e., has only finitely many v_x non-zero, then

$$\lim_{\Lambda \nearrow \mathrm{Z}} \big[\tilde{H}_\Lambda^{(r)}, a^*(v) \big] = a^*(\tilde{h}^{(r)} v),$$

where $\tilde{h}^{(r)}$is the bi-infinite Jacobi matrix defined on $\ell^2(\mathbf{Z})$ by

$$(\tilde{h}^{(r)} v)_x = \varepsilon_x v_x - \frac{1}{\Delta}(v_{x-1} + v_{x+1}). \tag{3.12}$$

For the finite system localized in $\Lambda = [a, b]$, we have

$$i \big[\tilde{H}_\Lambda^{(r)}, a^*(v) \big] = a^*(\tilde{h}_\Lambda^{(r)} v),$$

where $(\tilde{h}_\Lambda^{(r)} v)_x = (\tilde{h}^{(r)} v)_x$ for x in the bulk $[a+1, b-1]$, and at the boundary sites we get

$$\tilde{h}_\Lambda^{(r)} v_a = \varepsilon_a^+ v_a - \frac{1}{\Delta} v_{a+1}, \qquad \tilde{h}_\Lambda^{(r)} v_b = \varepsilon_b^- v_b - \frac{1}{\Delta} v_{b-1}.$$

Some important properties of $\tilde{h}^{(r)}$ were given in Section 2.1. Recall the existence of a zero-mode (property (ii)) given by:

$$v_{0,x} = \sin \theta_x^{(r)} = \frac{1}{\cosh(\eta(x - r))}.$$

That this is an eigenvector of $\tilde{h}^{(r)}$with eigenvalue zero, is easily verified:

$$\varepsilon_x^{\pm} v_{x,0} - \frac{1}{\Delta} v_{x\pm1,0} = \frac{\sin\theta_{x\pm1}^{(r)}}{\Delta \sin\theta_x^{(r)}} \sin\theta_x^{(r)} - \frac{1}{\Delta} \sin\theta_{x\pm1}^{(r)} = 0.$$

The origin of the zero-mode v_0 is well understood. It arises from the rotation symmetry of $H_{J,\Lambda}$. This can be seen from Lemma 3.2, by taking $\varphi \propto J^{-1/2}$ and formally identifying $J^{-1/2}\tilde{S}_x^-$ with a_x^* (this identification will be made more precise below). For every $N \in \mathbf{N}$, there is a zero-energy vector $\psi_{0,N}$ corresponding to an N-particle occupation of v_0:

$$\psi_{0,N} = \frac{1}{|v_0|_2^N \sqrt{N!}} a^*(v_0)^N \tilde{\Omega}.$$

We define P_0 the projection on v_0 and $\tilde{P}_0$ the projection onto the zero-energy vectors, i.e.,

$$P_0 = \frac{|v_0\rangle\langle v_0|}{|v_0|_2}, \tag{3.13}$$

$$\tilde{P}_0 = \bigoplus_{N\in\mathrm{N}} |\psi_{0,N}\rangle\langle\psi_{0,N}|. \tag{3.14}$$

For completeness we recall that we will always use the standard orthonormal basis in $\mathcal{F}$, i.e., the set $\{\varphi_{\vec{n}} \mid \vec{n} \in \mathcal{N}\}$, where

$$\varphi_{\vec{n}} = \prod_{x\in\mathrm{Z}} \frac{1}{(n_x!)^{1/2}} (a_x^*)^{n_x} \tilde{\Omega}. \tag{3.15}$$

We use the same symbol $\varphi_{\vec{n}}$ to denote a vector in the spin Hilbert space $\mathcal{H}_J$ and the boson space $\mathcal{F}$ since we will use the identification of $\mathcal{H}_J$ with a subspace of $\mathcal{F}$ as discussed before.

4. The large spin limit as a quantum central limit

Since the grand canonical states can be written as the 'all +' state for rotated spin operators, we are in the usual situation of a fully ferromagnetic state in which we expect a boson limit after rescaling with $J^{-1/2}$, i.e., $(1/\sqrt{2J})\tilde{S}_x^- \to a_x^*$ in some sense.

One way to make this precise is to define fluctuation operators: for $v_x \in \mathbf{R}^3$,

$$F_J(v_x) = \frac{1}{\sqrt{J}} \big(v_x \cdot S_x - \omega^{(r)}(v_x \cdot S_x) \big),$$

i.e., $F_J(v_x)$ measures the deviation from the ground state expectation value of the spin in the v_x direction. Similar fluctuation operators are used to study fluctuations of extensive observables, and their thermodynamic limit can be taken as a noncommutative central limit [8, 9].

A connection between the spin limit $J \to \infty$ and these quantum central limits was made in [15], with the caveat that each spin-J had to be represented as a sum of spin-(1/2)'s, instead of working with an irreducible representation. This latter restriction is however not necessary. We have, using results from [1, 14]:

$$\begin{aligned} \omega^{(r)}\big(\exp\{iv_x \cdot S_x\}\big) &= \Big\{ \cos\Big(\frac{1}{2}|v_x|\Big) + i\frac{v_x \cdot u_x^{(r)}}{|v|} \sin\Big(\frac{1}{2}|v_x|\Big)\Big\}^{2J}, \qquad (4.1)\\ \omega^{(r)}(v_x \cdot S_x) &= J(v_x \cdot u_x^{(r)}),\\ \omega^{(r)}\big(F_J(v_x)F_J(w_x)\big) &= \frac{1}{2}\big(v_x \cdot w_x - (v_x \cdot u_x^{(r)})(w_x \cdot u_x^{(r)}) + i(v_x \times w_x)\cdot u_x^{(r)}\big). \end{aligned}$$

The latter quantity defines a (degenerate) inner product on $\mathbf{R}^3$:

$$\langle v_x, w_x\rangle_x = 2\omega^{(r)}\big(F_J(v_x)F_J(w_x)\big). \qquad (4.2)$$

It is not hard to use (4.1) to show that

$$\lim_{J\to\infty} \omega^{(r)}\big(\exp\{iF_J(v_x)\}\big) = \exp\Big\{-\frac{1}{2}\langle v_x, v_x\rangle\Big\}. \qquad (4.3)$$

Clearly if either v_x or w_x is along the $u_x^{(r)}$-direction, then $\langle v_x, w_x\rangle_x = 0$. Hence (4.2) defines an inner product in $\mathbf{R}^2 = \mathbf{C}$, the tangent plane to the unit sphere at $u_x^{(r)}$. If for $\tilde{v}_x, \tilde{w}_x \in \mathbf{R}^2$, v_x, w_x are the corresponding vectors in $\mathbf{R}^3$ (see Section 3.2),

$$\langle \tilde{v}_x, \tilde{w}_x\rangle_x \equiv \langle v_x, w_x\rangle_x = \overline{(\tilde{v}_x^1 + i\tilde{v}_x^2)}(\tilde{w}_x^1 + i\tilde{w}_x^2),$$

i.e., the standard inner product in $\mathbf{C}$. We see that there are no fluctuations in the direction perpendicular to the tangent plane at the classical zero-energy solution. Two vectors in $\mathbf{R}^3$ at the same site will be called equivalent if their projection onto the tangent plane at $u_x^{(r)}$ is the same.

For $v = (v_x \in \mathbf{R}^3)_{x\in\mathrm{Z}}$, with only finitely many $v_x \neq 0$, we simply extend this by putting

$$\begin{aligned} F_J(v) &= \sum_{x\in\mathrm{Z}} F_J(v_x),\\ \langle v, w\rangle &= 2\omega\big(F_J(v)F_J(w)\big) = \sum_{x\in\mathrm{Z}} \langle v_x, w_x\rangle_x. \end{aligned}$$

Using (4.3), and standard techniques, the quantum central limit theorem follows:

$$\lim_{J\to\infty} \omega^{(r)}\big(\exp\{iF_J(v_1)\}\dots\exp\{iF_J(v_n)\}\big) = \tilde{\omega}\big(W(v_1)\dots W(v_n)\big),$$

where $\tilde{\omega}$ is the Fock state on the CCR-algebra $\mathrm{CCR}(\ell^2(\mathbf{Z}),\sigma)$ introduced in Section 3.3, and the vectors $v_1,\dots,v_n$ on the r.h.s. mean their respective equivalence classes in $\ell^2(\mathbf{Z})$. The result can be understood intuitively from

$$v_x\cdot S_x=\tilde{v}_x^1\tilde{S}_x^1+\tilde{v}_x^2\tilde{S}_x^2=(\tilde{v}_x^1-i\tilde{v}_x^2)\tilde{S}_x^+ +(\tilde{v}_x^1+i\tilde{v}_x^2)\tilde{S}_x^-$$

for $\tilde{v}_x\in\mathbf{R}^2$ and corresponding $v_x\in\mathbf{R}^3$.

When studying properties of the GNS Hamiltonian, it is actually easier to make a correspondence between the GNS Hilbert spaces $\mathcal{H}_J$ of $\omega^{(r)}$ and $\mathcal{F}$ of $\tilde{\omega}$.

Following [22], introduce the projection $P_{J,x}$ on $\mathcal{F}$ which projects onto the first $2J+1$ Boson states at site x, i.e., on the states $\varphi_{\vec{n}}$ (3.15) with $0\le n_x\le 2J$, and denote $P_J=\prod_x P_{J,x}$, i.e., P_J projects onto the states $\varphi_{\vec{n}}$ with $\vec{n}\in\mathcal{N}_J$, see (3.10). By identifying $\varphi_{\vec{n}}$ (3.11) with $\varphi_{\vec{n}}$ (3.15), it is clear that $\mathcal{H}_J=P_J\mathcal{F}$, where = means unitarily equivalent. Under this equivalence, we find that the spin operators are given by [22]:

$$\frac{1}{\sqrt{2J}}\tilde{S}_x^-=P_Ja_x^*g_J(x)^{1/2},\quad \frac{1}{\sqrt{2J}}\tilde{S}_x^+=g_J(x)^{1/2}a_xP_J,\quad J-\tilde{S}_x^3=P_Ja_x^*a_xP_J, \tag{4.4}$$

where $g_J(x)=g_J(a_x^*a_x)$ and

$$g_J(n)=\begin{cases}1-\frac{1}{2J}n, & n\le 2J,\\ 0, & n>2J.\end{cases}$$

5. The low energy spectrum

5.1. Some estimates for the Hamiltonian

We first need the following identities:

Lemma 5.1.

$$\cos\theta_{x-1}^{(r)}+\cos\theta_{x+1}^{(r)}=\varepsilon_x\cos\theta_x^{(r)}, \tag{5.1}$$

$$\Delta^{-1}(\sin\theta_{x-1}^{(r)}+\sin\theta_{x+1}^{(r)})=\varepsilon_x\sin\theta_x^{(r)}, \tag{5.2}$$

$$\sin\theta_x^{(r)}\sin\theta_{x\pm1}^{(r)}+\Delta^{-1}\cos\theta_x^{(r)}\cos\theta_{x\pm1}^{(r)}=\Delta^{-1}, \tag{5.3}$$

$$\Delta^{-1}\cos\theta_x^{(r)}\sin\theta_{x\pm1}^{(r)}-\sin\theta_x^{(r)}\cos\theta_{x\pm1}^{(r)}=\mp\sqrt{1-\Delta^{-2}}\sin\theta_x^{(r)}, \tag{5.4}$$

$$\frac{1}{\Delta}\sin\theta_{x\pm1}^{(r)}\sin\theta_x^{(r)}+\cos\theta_{x\pm1}^{(r)}\cos\theta_x^{(r)}\mp\sqrt{1-\Delta^{-2}}\cos\theta_x^{(r)}=\varepsilon_x^{\pm}. \tag{5.5}$$

Proof. These are straightforward computations using the definitions (3.2) and (3.3) of $\cos\theta_x^{(r)}$ and $\sin\theta_x^{(r)}$, and the addition laws for sinh and cosh. □

With this lemma we can write the Hamiltonian in terms of the $\tilde{S}$-operators.

Corollary 5.1.

$$\begin{aligned}H^J_{x,x+1} = J^2 &- \frac{1}{2\Delta}\big(\tilde{S}^+_x\tilde{S}^-_{x+1} + \tilde{S}^-_x\tilde{S}^+_{x+1}\big) - \gamma_{x,x+1}\tilde{S}^3_x\tilde{S}^3_{x+1}\\ &+ J\sqrt{1-\Delta^{-2}}\big(\cos\theta^{(r)}_x\tilde{S}^3_x - \cos\theta^{(r)}_{x+1}\tilde{S}^3_{x+1}\big)\\ &+ \sqrt{1-\Delta^{-2}}\big(\sin\theta^{(r)}_x\tilde{S}^1_x\tilde{S}^3_{x+1} - \sin\theta^{(r)}_{x+1}\tilde{S}^3_x\tilde{S}^1_{x+1}\big)\\ &- J\sqrt{1-\Delta^{-2}}\big(\sin\theta^{(r)}_x\tilde{S}^1_x - \sin\theta^{(r)}_{x+1}\tilde{S}^1_{x+1}\big),\end{aligned}$$

where

$$\gamma_{x,x+1} = \varepsilon^+_x + \sqrt{1-\Delta^{-2}}\cos\theta^{(r)}_x = \varepsilon^-_{x+1} - \sqrt{1-\Delta^{-2}}\cos\theta^{(r)}_{x+1}.$$

Proof. This follows immediately from the relations (3.6)–(3.8), and the previous lemma. □

With the Hamiltonian in terms of the $\tilde{S}$-operators, we can apply the unitary transformation (4.4) to write the spin Hamiltonian as an operator on $\mathcal{F}$:

$$\begin{aligned}\frac{1}{J}H_{J,\Lambda} = P_J\Big\{&\sum_{x=a}^{b-1}\big[\varepsilon^+_x g_J(x)a^*_x a_x - \Delta^{-1}a^*_{x+1}g_J(x+1)^{1/2}g_J(x)^{1/2}a_x\big]\\ &+\sum_{x=a+1}^{b}\big[\varepsilon^-_x g_J(x)a^*_x a_x - \Delta^{-1}a^*_{x-1}g_J(x-1)^{1/2}g_J(x)^{1/2}a_x\big]\\ &+\frac{\sqrt{1-\Delta^{-2}}}{2J}\big[\cos\theta^{(r)}_b N_b^2 - \cos\theta^{(r)}_a N_a^2\big] + \sum_{x=a}^{b-1}\gamma_{x,x+1}\frac{(N_x - N_{x+1})^2}{2J}\\ &+\frac{\sqrt{1-\Delta^{-2}}}{2J^{1/2}}\sum_{x=a}^{b-1}\Big[\sin\theta^{(r)}_{x+1}(g_J(x+1)^{1/2}a_{x+1} + a^*_{x+1}g_J(x+1)^{1/2})N_x\\ &-\sin\theta^{(r)}_x(g_J(x)^{\frac{1}{2}}a_x + a^*_x g_J(x)^{1/2})N_{x+1}\Big]\Big\}P_J,\end{aligned} \tag{5.6}$$

and likewise for the ∞-volume GNS Hamiltonian:

$$\begin{aligned}\frac{1}{J}H_J = P_J\Big\{&\sum_{x\in\mathbb{Z}}\big[\varepsilon_x g_J(x)a^*_x a_x\\ &-\Delta^{-1}a^*_x g_J(x)^{1/2}\big(g_J(x-1)^{1/2}a_{x-1} + g_J(x+1)^{1/2}a_{x+1}\big)\big]\\ &+\sum_{x=a}^{b-1}\gamma_{x,x+1}\frac{(N_x - N_{x+1})^2}{2J}\\ &+\frac{\sqrt{1-\Delta^{-2}}}{2J^{1/2}}\sum_{x\in\mathbb{Z}}\sin\theta^{(r)}_x(g_J(x)^{1/2}a_x + a^*_x g_J(x)^{1/2})(N_{x-1} - N_{x+1})\Big\}P_J.\end{aligned} \tag{5.7}$$

In the usual language of spin wave theory [6, 7, 22], the first term (between $[\dots]$) in the Hamiltonian is called the kinematical interaction $H_{J,\mathrm{kin}}$, and the second term the dynamical interaction $H_{J,\mathrm{dyn}}$. The last term, which describes transitions between subspaces with constant number of particles, is not usually present. We denote it $H_{J,\mathrm{tran}}$. Note that we define these three operators with the right scaling already included, i.e.,

$$\frac{1}{J}H_J = P_J\{H_{J,\mathrm{kin}} + H_{J,\mathrm{dyn}} + H_{J,\mathrm{tran}}\}P_J.$$

We will only let these operators act on vectors in $\mathcal{H}_J$, hence we may forget about the P_J. To further simplify some notation, introduce

- the column vector A of annihilation operators:

$$A = \begin{pmatrix} \vdots \\ a_x \\ \vdots \end{pmatrix},$$

- the diagonal matrix G_J:

$$G_J(x, y) = g_J(x)\delta_{x,y}.$$

Then we can write

$$\begin{aligned} H_{J,\mathrm{kin}} &= A^* G_J^{1/2} \tilde{h}^{(r)} G_J^{1/2} A - \frac{1}{2J}\sum_x \varepsilon_x a_x^* a_x, \\ \tilde{H}^{(r)} &= A^* \tilde{h}^{(r)} A, \end{aligned}$$

where $\tilde{h}^{(r)}$ is the one-particle Boson Hamiltonian, see (3.12), i.e., the matrix with entries

$$\tilde{h}^{(r)}(x, y) = \varepsilon_x \delta_{x,y} - \Delta^{-1}(\delta_{x-1,y} + \delta_{x+1,y}).$$

In the following we will fix for every J an $n_J \in \mathbf{N}$ with $0 < n_J < J$ and make statements about the subspace $P_{n_J}\mathcal{H}_J$ of $\mathcal{H}_J$. In the end we will formulate results on the whole of $\mathcal{H}_J$ by adding an external pinning field which will take care of the states in $(P_{n_J}\mathcal{H}_J)^\perp \cap \mathcal{H}_J$. For simplicity denote $\mathcal{H}_{n_J} = P_{n_J}\mathcal{H}_J$.

Lemma 5.2. *On $\mathcal{H}_{n_J}$ we have the lower bound*

$$H_{J,\mathrm{kin}} \geq \Big(\tilde{\gamma}^{(r)} g_J(2n_J) - \frac{1}{J}\Big)N_{\mathrm{tot}} - \tilde{\gamma}^{(r)} A^* G^{1/2} P_0 G^{1/2} A,$$

where $\tilde{\gamma}^{(r)}$ is the spectral gap of $\tilde{h}^{(r)}$ and P_0 is defined in (3.13). *An upper bound (on the whole $\mathcal{H}_J$) is given by*

$$H_{J,\mathrm{kin}} \leq \|\tilde{h}^{(r)}\| N_{\mathrm{tot}}.$$

Proof. P_{n_J} projects onto the vectors with at most $2n_J$ particles per site, such that on $\mathcal{H}_{n_J}$:

$$G_J \geq g_J(2n_J)\mathbf{1}.$$

Obviously $G_J \leq \mathbf{1}$ on $\mathcal{H}_J$. The lemma follows from the bounds on $\tilde{h}^{(r)}$:

$$\tilde{\gamma}^{(r)}(\mathbf{1} - P_0) \leq \tilde{h}^{(r)} \leq \|\tilde{h}^{(r)}\|\mathbf{1}$$

and also $\varepsilon_x \leq 2$. □

We are going to compare the spectrum of $H_{J,\mathrm{kin}}$ with the spectrum of $\tilde{H}^{(r)}$. Both operators commute with N_{tot} so it is sufficient to compare them on eigenstates of N_{tot}. Also for $H_{J,\mathrm{dyn}}$ it is sufficient to look at eigenstates of N_{tot}.

In the proof of the following lemmata we will use the following notation: for $\vec{n}, \vec{m} \in \mathcal{N}$:

$$T_x^{\pm}\vec{n} = \vec{m} \quad \text{iff} \quad \begin{cases} m_x = n_x + 1, \\ m_{x\pm 1} = n_{x\pm 1} - 1, \\ m_y = n_y, \quad \forall y \neq x, x+1, \end{cases}$$

$$A_x^{\pm}\vec{n} = \vec{m} \quad \text{iff} \quad \begin{cases} m_x = n_x \pm 1, \\ m_y = n_y, \quad \forall y \neq x. \end{cases}$$

Lemma 5.3. *Let $\psi_N \in \mathcal{H}_{n_J}$, $\|\psi_N\| = 1$, $N_{\mathrm{tot}}\psi_N = N\psi_N$. Then*

$$\big\|(\tilde{H}^{(r)} - H_{J,\mathrm{kin}})\psi_N\big\| \leq 2(1 + \Delta^{-1})\frac{n_J N}{J},$$
$$\big\|H_{J,\mathrm{dyn}}\psi_N\big\| \leq \frac{4n_J N}{J}.$$

Proof. We can write

$$\psi_N = \sum_{\vec{n}} c_{\vec{n}}\,\varphi_{\vec{n}} \quad \text{with} \quad \sum_{\vec{n}} |c_{\vec{n}}|^2 = 1,$$

where the sum runs over $\vec{n} \in \mathcal{N}_J$ for which $\sum_x n_x = N$. On basis vectors, we have

$$\begin{aligned} H_{J,\mathrm{kin}}\varphi_{\vec{n}} = {} & \sum_x \varepsilon_x n_x g_J(n_x)\varphi_{\vec{n}} \\ & - \Delta^{-1}\sum_x \big[(n_x+1)g_J(n_x)g_J(n_{x-1}-1)n_{x-1}\big]^{1/2}\varphi_{T_x^-\vec{n}} \\ & - \Delta^{-1}\sum_x \big[(n_x+1)g_J(n_x)g_J(n_{x+1}-1)n_{x+1}\big]^{1/2}\varphi_{T_x^+\vec{n}} \end{aligned}$$

and for $\tilde{H}^{(r)}$ the same with the $g_J \equiv 1$. We compare term by term. The first one gives:

$$\Big\| \sum_{\vec{n},x} c_{\vec{n}} \varepsilon_x \big[n_x - n_x g_J(n_x)\big] \varphi_{\vec{n}} \Big\|^2 = \frac{1}{(2J)^2} \sum_{\vec{n}} |c_{\vec{n}}|^2 \Big| \sum_x \varepsilon_x n_x^2 \Big|^2 \leq \frac{4N^2 n_J^2}{J^2},$$

where we used $\varepsilon_x \leq 2$, $n_x \leq 2n_J$ and $\sum_x n_x = N$. For the second term we use analogously $g_J(n_x) \geq g_J(2n_J)$, and find:

$$\begin{aligned}
&\Big\| \sum_{\vec{n},x} c_{\vec{n}} \big([(n_x+1)n_{x-1}]^{1/2} - [(n_x+1)g_J(n_x)g_J(n_{x-1}-1)n_{x-1}]^{1/2}\big) \varphi_{T_x^- \vec{n}} \Big\| \\
&= \Big\| \sum_{\vec{m},x} c_{T_{x-1}^+ \vec{m}} \big([m_x(m_{x-1}+1)]^{1/2} \\
&\quad - [m_x g_J(m_x - 1) g_J(m_{x-1})(m_{x-1}+1)]^{1/2}\big) \varphi_{\vec{m}} \Big\| \\
&\leq \sum_{\vec{m}} \Big(\sum_x |c_{T_{x-1}^+ \vec{m}}| \, \big| [m_x(m_{x-1}+1)]^{1/2} \\
&\quad - [m_x g_J(m_x - 1) g_J(m_{x-1})(m_{x-1}+1)]^{1/2} \big| \Big)^2 \\
&\leq \sum_{\vec{m}} \Big(\sum_x |c_{T_{x-1}^+ \vec{m}}| [m_x(m_{x-1}+1)]^{1/2} (1 - g_J(2n_J)) \Big)^2 \\
&\leq \frac{n_J^2}{J^2} \sum_{\vec{m}} \Big(\sum_x |c_{T_{x-1}^+ \vec{m}}|^2 (m_{x-1}+1) \Big) \Big(\sum_x m_x \Big) = \frac{N^2 n_J^2}{J^2}
\end{aligned}$$

and the same for the third term. Summing everything together we find

$$\big\| (\tilde{H}^{(r)} - H_{J,\mathrm{kin}}) \psi_N \big\| \leq 2(1 + \Delta^{-1}) \frac{N n_J}{J}.$$

For $H_{J,\mathrm{dyn}}$ we use the same reasoning, $\gamma_{x,x+1} \leq 1$, and

$$(n_x - n_{x+1})^2 \leq 2n_x^2 + 2n_{x+1}^2 \leq 4n_J(n_x + n_{x+1})$$

to find

$$\big\| H_{J,\mathrm{dyn}} \psi_N \big\| \leq \frac{4 n_J N}{J}.$$

□

For $H_{J,\mathrm{tran}}$ we have the following estimate.

Lemma 5.4. *Let $\psi \in \mathcal{H}_{n_J}$, $\|\psi\| = 1$. Then*

$$\big\| H_{J,\mathrm{tran}} \psi \big\| \leq 2\sqrt{1 - \Delta^{-2}} |v_0|_1 \Big(\frac{(2n_J+1)(4n_J)^2}{J} \Big)^{1/2},$$

where $|v_0|_1$ is the ℓ^1-norm of the zero-mode.

Proof. Let

$$\psi = \sum_{\vec{n}} c_{\vec{n}} \varphi_{\vec{n}}.$$

Then

$$\begin{aligned}
&\Big\| \sum_x \sin\theta_x^{(r)} g_J(x)^{1/2} a_x (N_{x-1} - N_{x+1})\psi \Big\|^2 \\
&\quad = \Big\| \sum_{\vec{n},x} c_{\vec{n}} \sin\theta_x^{(r)} g_J(n_x - 1)^{1/2} n_x^{1/2} (n_{x-1} - n_{x+1}) \varphi_{A_x^- \vec{n}} \Big\|^2 \\
&\quad = \Big\| \sum_{\vec{n},x} c_{A_x^+ \vec{n}} \sin\theta_x^{(r)} g_J(n_x)^{1/2} (n_x + 1)^{1/2} (n_{x-1} - n_{x+1}) \varphi_{\vec{n}} \Big\|^2 \\
&\quad = \sum_{\vec{n}} \Big| \sum_x c_{A_x^+ \vec{n}} \sin\theta_x^{(r)} g_J(n_x)^{1/2} (n_x + 1)^{1/2} (n_{x-1} - n_{x+1}) \Big|^2 \\
&\quad \le \sum_{\vec{n}} \Big(\sum_x \sin\theta_x^{(r)} |c_{A_x^+ \vec{n}}|^2 \Big) \Big(\sum_x \sin\theta_x^{(r)} (n_x + 1) |n_{x-1} - n_{x+1}|^2 \Big) \\
&\quad \le |v_0|_1 (2n_J + 1)(4n_J)^2 \sum_{\vec{n}} \sum_x \sin\theta_x^{(r)} |c_{A_x^+ \vec{n}}|^2 \\
&\quad = |v_0|_1^2 (2n_J + 1)(4n_J)^2
\end{aligned}$$

and likewise for the second term. □

5.2. Strong convergence and spectral concentration

Recall the following definitions. For A a self-adjoint operator and $a, b \in \mathbf{R}$, denote by $P_{(a,b)}(A)$ the spectral projection of A onto (a, b). For A acting on Fock space, denote

- the spectrum of A in $\mathcal{F}$ by $\sigma(A)$;
- the spectrum of $P_J A P_J$ in $\mathcal{H}_J$ by $\sigma_J(A)$;
- the spectrum of $P_{n_J} A P_{n_J}$ in $\mathcal{H}_{n_J}$ by $\sigma_{n_J}(A)$.

Also denote by s-lim the strong, or strong resolvent, operator limit for bounded, resp. unbounded operators acting on $\mathcal{F}$.

In the GNS space $\mathcal{H}_J$ it is convenient to define S^3_{tot} in the renormalized sense: $S^3_{\text{tot}} = \sum_{x \in \mathrm{Z}} \big[S^3_x - \operatorname{sgn}(x - 1/2) \big]$. Also denote

$$\mu = \sum_{x \in \mathrm{Z}} \Big[\cos\theta_x^{(r)} - \operatorname{sgn}\Big(x - \frac{1}{2}\Big) \Big],$$

i.e., μ is the 3-magnetization of a classical ground state $\{(\theta_x^{(r)}, \varphi)\}_{x \in \mathrm{Z}}$ (see Section 3.1).

Proposition 5.1. *We have*

$$\operatorname*{s-lim}_{J\to\infty} \frac{1}{J} H_J = \tilde{H}^{(r)},$$
$$\operatorname*{s-lim}_{J\to\infty} \frac{1}{J} S^3_{\mathrm{tot}} = \mu \mathbf{1}.$$

Proof. Introduce the set $\mathcal{D}$, the finite linear space of vectors $\varphi_{\vec{n}}$ with $\vec{n} \in \mathcal{N}$. $\mathcal{D}$ is a common core for $(1/J)H_J$, for all J, and $\tilde{H}^{(r)}$.

Take $\psi \in \mathcal{D}$ arbitrary (but normalized for simplicity) and denote

$$\psi = \sum_{\vec{n}} c_{\vec{n}} \varphi_{\vec{n}},$$
$$N_\psi = \sup_{\vec{n}:\ c_{\vec{n}} \neq 0} \sum_x n_x.$$

Note that by assumption, ψ is a finite sum of $\varphi_{\vec{n}}$ with $\sum_x n_x < \infty$, and hence also $N_\psi < \infty$.

Now take J large enough such that $\psi \in \mathcal{H}_J$. From the proof of Lemma 5.3 and 5.4 it is clear that we can use $2n_J \leq N_\psi$, as soon as $2J > N_\psi$, hence

$$\Big\| \Big(\frac{1}{J} H_J - \tilde{H}^{(r)} \Big) \psi \Big\| \leq (3 + \Delta^{-1}) \frac{N_\psi^2}{J} + 2\sqrt{1 - \Delta^{-2}} |v_0|_1 \Big(\frac{(N_\psi + 1)(2N_\psi)^2}{J} \Big)^{1/2}.$$

The first result follows from [20, Theorem VIII.25 (a)].

To prove the second statement, write

$$\begin{aligned} S^3_x &= \cos\theta^{(r)}_x \tilde{S}^3_x - \sin\theta^{(r)}_x \tilde{S}^1_x \\ &= P_J \big\{ \cos\theta^{(r)}_x (J - N_x) - \sin\theta^{(r)}_x \big(g_J(x)^{1/2} a_x + a^*_x g_J(x)^{1/2} \big) \big\} P_J \end{aligned}$$

and hence

$$\begin{aligned} S^3_{\mathrm{tot}} &= \sum_x S^3_x - \operatorname{sgn}\Big(x - \frac{1}{2} \Big) \\ &= P_J \Big\{ \mu J - \sum_x \cos\theta^{(r)}_x N_x - \sum_x \sin\theta^{(r)}_x \big(g_J(x)^{1/2} a_x + a^*_x g_J(x)^{1/2} \big) \Big\} P_J. \end{aligned}$$

Clearly $\sum_x \cos\theta^{(r)}_x N_x \leq N_{\mathrm{tot}}$, and for $\psi \in \mathcal{D}$ as before

$$\begin{aligned} &\Big\| \sum_x \sin\theta^{(r)}_x a^*_x g_J(x)^{1/2} \big(\sum_{\vec{n}} c_{\vec{n}} \varphi_{\vec{n}} \big) \Big\|^2 \\ &\qquad = \Big\| \sum_{\vec{n},x} \sin\theta^{(r)}_x c_{\vec{n}} (n_x + 1)^{1/2} g_J(n_x)^{1/2} \varphi_{A_x \vec{n}} \Big\|^2 \leq (N_\psi + 1) |v_0|_1^2. \end{aligned}$$

Hence

$$\left\| \Big(\frac{1}{J} S^3_{\rm tot} - \mu\Big)\psi \right\| \leq \frac{1}{J}\big(N_\psi + 2(N_\psi + 1)^{1/2} |v_0|_1\big)$$

and $(1/J)S^3_{\rm tot} \to \mu\mathbf{1}$ strongly on $\mathcal{F}$. □

Corollary 5.2.

(i) *If* $\lambda \in \sigma(\tilde{H}^{(r)})$*, there exists* $\lambda_J \in \sigma_J((1/J)H_J)$ *such that*

$$\lim_{J\to\infty} \lambda_J = \lambda.$$

(ii) *If* $a, b \in \mathbf{R}$*, and* $a, b \notin \sigma_{pp}(\tilde{H}^{(r)})$*, then*

$$\operatorname*{s-lim}_{J\to\infty} P_{(a,b)}\Big(\frac{1}{J}H_J\Big) = P_{(a,b)}\big(\tilde{H}^{(r)}\big).$$

(iii) *If* $a, b \in \mathbf{R}$*, and* $\mu \in (a, b)$*, then*

$$\operatorname*{s-lim}_{J\to\infty} P_{(a,b)}\Big(\frac{1}{J}S^3_{\rm tot}\Big) = \mathbf{1}.$$

Proof. The previous proposition and [20, Theorem VIII.24]. □

In addition we can prove spectral concentration of $(1/J)H_J$ around discrete eigenvalues of $\tilde{H}^{(r)}$.

Proposition 5.2. *For every isolated eigenvalue* E *of* $\tilde{H}^{(r)}$*, there exists an interval*

$$I_J = (E - \varepsilon_J, E + \varepsilon_J) \quad \text{with} \quad \lim_{J\to\infty} \varepsilon_J \frac{J}{\ln J} = 0,$$

such that for any interval I *around* E *s.t.* $I \cap \sigma(\tilde{H}^{(r)}) = \{E\}$*:*

$$\operatorname*{s-lim}_{J\to\infty} P_{I\setminus I_J}\Big(\frac{1}{J}H_J\Big) = 0,$$
$$\operatorname*{s-lim}_{J\to\infty} P_{I_J}\Big(\frac{1}{J}H_J\Big) = P_{\{E\}}\big(\tilde{H}^{(r)}\big).$$

Proof. Let ψ_E be the simultaneous eigenvector of $\tilde{H}^{(r)}$ with eigenvalue E, and of $N_{\rm tot}$ with eigenvalue N_E. It follows that

$$N_E \leq \frac{E}{\tilde{\gamma}^{(r)}}.$$

We have from Lemma 5.3

$$\left\|\left(\frac{1}{J}H_J - E\right)\psi_E\right\| \leq \frac{c(\Delta, r, E)}{J}$$

or,

$$\lim_{J\to\infty} \frac{J}{\ln J}\left\|\left(\frac{1}{J}H_J - E\right)\psi_E\right\| = 0.$$

Hence it follows that E is a first order pseudo-eigenvalue with first-order pseudo eigenvector ψ_E, and the result follows from [20, Theorem XII.22]. □

Remark 5.1. Applying the same reasoning to S^3_{tot}, we find that the interval (a, b) in item (iii) of Corollary 5.2 can be chosen as

$$(a, b) = (\mu - \varepsilon_J, \mu + \varepsilon_J)$$

with again $\lim_J \varepsilon_J J(\ln J)^{-1} = 0$.

5.3. Convergence of the spectrum with a pinning field

To prove full convergence of the spectrum, we have to add the external field (2.4) to H_J, or, to have a positive operator, add:

$$\sup_x \left(|h_{J,x}|J - \vec{h}_{J,x}\cdot\vec{S}_x\right) = h_J \sup_x \left(J - \tilde{S}^3_x\right) = h_J \sup_x (N_x) \qquad (5.8)$$

with $h_J > 0$, see also the end of Section 3.2, and $N_x = a^*_x a_x$. Let us assume we add this field to $(1/J)H^J$, so h_J already contains the factor J^{-1}.

Take $0 < n_J < J$ as before, we get that on $(\mathcal{H}_{n_J})^\perp \cap \mathcal{H}_J$

$$\frac{1}{J}H_J + h_J \sup_x N_x \geq h_J \sup_x N_x \geq h_J n_J \mathbf{1}.$$

Clearly, by choosing h_J such that

$$\lim_{J\to\infty} h_J n_J = \infty$$

statements about the spectrum on $\mathcal{H}_J$ reduce to statements about the spectrum on $\mathcal{H}_{n_J}$. Or, if one chooses to make statements about the spectrum below a certain value E, it is sufficient to choose h_J such that $\lim_J h_J n_J > E + \varepsilon$.

Convergence of the spectrum of $H_{J,\mathrm{kin}} + H_{J,\mathrm{dyn}}$ can be proved under the weakest assumptions on n_J. First we prove convergence of the spectrum of $H_{J,\mathrm{kin}}$.

Proposition 5.3. *Let $n_J = [J(\ln J)^{-1}]$, where $[\cdot]$ denotes the integer part. If $\lambda \notin \sigma(\tilde{H}^{(r)})$, then $\lambda \notin \sigma_{n_J}(H_{J,\mathrm{kin}})$ for J large enough.*

Proof. Assume that $\lambda \in \sigma_{n_J}(H_{J,\text{kin}})$ for all J larger than some J_0. Take $\delta > 0$ arbitrary, and $\psi_J \in \mathcal{H}_{n_J}$ a normalized approximate eigenvector:

$$\left\|(H_{J,\text{kin}} - \lambda)\psi_J\right\| < \delta.$$

More precisely, take ψ_J such that

$$P_{(\lambda-\delta,\lambda+\delta)}(H_{J,\text{kin}})\psi_J = \psi_J.$$

Since $[H_{J,\text{kin}}, N_{\text{tot}}] = 0$ we can take ψ_J an eigenstate of N_{tot} with eigenvalue N_J. Since ψ_J is orthogonal to the ground state space, it follows from Lemma 5.2 that

$$N_J \leq \frac{\lambda + \delta}{\tilde{\gamma}^{(r)} g_J(2n_J) - J^{-1}}.$$

By our choice of n_J, we have $\lim_J n_J J^{-1} = 0$, or $\lim_J g_J(2n_J) = 1$, and for J large enough,

$$N_J \leq \frac{2(\lambda + \delta)}{\tilde{\gamma}^{(r)}}.$$

Putting this into the bounds of Lemma 5.3, we find

$$\begin{aligned}\left\|(\tilde{H}^{(r)} - \lambda)\psi_J\right\| &\leq \left\|(\tilde{H}^{(r)} - H_{J,\text{kin}})\psi_J\right\| + \left\|(H_{J,\text{kin}} - \lambda)\psi_J\right\| \\ &\leq \frac{4(1 + \Delta^{-1})(\lambda + \delta)}{\tilde{\gamma}^{(r)}} \frac{n_J}{J} + \delta\end{aligned}$$

and it follows that ψ_J is an approximate eigenvector for $\tilde{H}^{(r)}$ as well and $\lambda \in \sigma(\tilde{H}^{(r)})$. □

Now we add $H_{J,\text{dyn}}$:

Proposition 5.4. *Let again* $n_J = [J(\ln J)^{-1}]$. *If* $\lambda \notin \sigma(\tilde{H}^{(r)})$, *then* $\lambda \notin \sigma_{n_J}(H_{J,\text{kin}} + H_{J,\text{dyn}})$ *for* J *large enough.*

Proof. Since $H_{J,\text{dyn}} \geq 0$, it follows that an approximate eigenvector ψ_J for $H_{J,\text{kin}} + H_{J,\text{dyn}}$ must satisfy

$$\left\|H_{J,\text{kin}}\psi_J\right\| \leq \lambda + \delta$$

with the same notation as in the previous proposition. Hence we get the same estimate on N_J as before, but this implies by Lemma 5.3 that

$$\lim_J \|H_{J,\text{dyn}}\psi_J\| = 0$$

and ψ_J is an approximate eigenvector for $\tilde{H}^{(r)}$ as well. □

Alternatively, these propositions prove that the spectrum of $H_{J,\mathrm{kin}}+H_{J,\mathrm{dyn}}+h_J \sup_x N_x$ converges to the spectrum of $\tilde{H}^{(r)}$, provided

$$\lim_J h_J = 0, \qquad \lim_J \frac{h_J J}{\ln J} = \infty.$$

To add $H_{J,\mathrm{tran}}$ we clearly have to relax our condition on n_J.

Proposition 5.5. *Let $n_J = [(J(\ln J)^{-1})^{1/3}]$. If $\lambda \notin \sigma(\tilde{H}^{(r)})$, then for J large enough, $\lambda \notin \sigma_{n_J}((1/J)H_J)$.*

Proof. By Lemma 5.4 we have for all $\psi \in \mathcal{H}_{n_J}$

$$\|H_{J,\mathrm{tran}}\psi\| \leq 2\sqrt{1-\Delta^{-2}}|v_0|_1 \Big(\frac{(2n_J+1)(4n_J)^2}{J}\Big)^{1/2} \|\psi\|$$

and by assumption the r.h.s. goes to 0 as $J \to \infty$. □

Hence we get convergence of the spectrum of $(1/J)H_J + h_J \sup_x N_x$ if

$$\lim_{J\to\infty} h_J = 0, \qquad \lim_{J\to\infty} h_J \Big(\frac{J}{\ln J}\Big)^{1/3} = \infty.$$

The results obtained in this section include the statements made in Section 2.3.

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