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Mathematical structure of magnons in quantum ferromagnets

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Abstract. We provide the mathematical structure and a simple, transparent and rigorous derivation of the magnons as elementary quasi-particle excitations at low temperatures and in the infinite spin limit for a large class of Heisenberg ferromagnets. The magnon canonical variables are obtained as fluctuation operators in the infinite spin limit. Their quantum character is governed by the size of the magnetization.

1. Introduction

The appearance of spin waves in quantum ferromagnets at low temperatures is one of the most basic physical quantum characteristics of quantum spin systems. It amounts to a boson representation of the low-temperature elementary excitations of a spin system. The basic steps in the understanding of this phenomenon were made by Bloch [1], Holstein–Primakoff [2], van Kramendonk–van Vleck [3] and, in the more technical work, by Dyson [4, 5].

From the point of view of mathematical physics, one discovers rigorous spin wave properties at regular times as upper or lower bounds of correlations for low temperature or ground states (see e.g. [6]). Undoubtly, the so-called Bethe ansatz [7] is the most representative low-temperature model of the spin wave theory, and, as is well known, to prove or disprove that the Bethe ansatz is correct for some models is presently a serious activity in mathematical physics. On the other hand, for a long time the spin wave theory of Holstein–Primakoff called for a simple, transparent and mathematically rigorous setting, in the sense that the conditions for the appearance of spin waves are clearly formulated, and that then the derivation of the spin waves or magnons is rigorously obtained.

As far as we know, the most serious attempt to achieve this has been made in [8]. On the basis of a classical domination principle which makes clear how to create the situation of a unique ferromagnetic quantum ground state, these authors define a 'physical' Hamiltonian, which is of direct relevance to the analysis of the Holstein–Primakoff and the Dyson formalism. They construct a 'quadratic boson Hamiltonian' in configuration space, and modulo an approximation, which is usually called plausible for large spins. They are able to obtain satisfying results on the asymptotic equality of the free energies of the quadratic boson Hamiltonian and the Heisenberg model in any dimension for low enough temperatures.

Here, stimulated by [8], we are able to clarify a number of aspects of the spin wave or magnon theory which were still absent. As in [8], we limit ourself to ferromagnets. Our

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main results are that we are able to use mathematical results on non-commutative central limit theorems in order to scrutinize the large spin limit correctly and to give a rigorous scheme for the formation of bosons. We are able to perform this programme without any uncontrollable approximation. The result is that the magnon canonical variables are nothing but fluctuation operators, whose mathematical structure is developed in [9, 10]. We rigorously perform the infinite spin limit, and prove that the Heisenberg model at low temperature becomes a system of non-interacting magnons. We discuss the quantum character of the magnons as a function of the magnetization at low temperatures.

Our results are on the level of the equilibrium states going beyond the study of its thermodynamic properties. We create the right conditions in order that the Heisenberg model system converges to a system of magnons, i.e. we give a rigorous mathematical meaning to the notion of 'magnon limit'. All our conditions are in the direction of enough ferromagnetism. This condition turns out to be the basic technical property to obtain Gaussian quantum fluctuations, which in turn is responsible for the linearization in the magnon limit. We are convinced that results can also be derived for antiferromagnets along the lines of what we have here if one is able to pin the suitable conditions on the interaction constants in order to get a specific antiferromagnetism. One may also ask the question whether analogous results are to be expected in the non-equilibrium case. With our present understanding, the situation is not clear. For situations near to equilibrium, as well as for metastable states, the magnon limit seems to work. However, in the situation far from equilibrium the problem is less clear.

2. Low-temperature magnons

At each site x of a finite domain Λ of a cubic lattice $\mathbb{Z}^\nu$, consider the spin-$(2S+1)$ variables $S^1(x), S^2(x), S^3(x)$, in the representation on $\otimes_{i=-S}^{S}(\mathbb{C}^2)_i$ given by

$$S^\mu(x) = \sum_{i=-S}^{S} \sigma_i^\mu(x) \tag{1}$$

$\mu = 1, 2, 3$, the $\sigma_i^\mu(x)$ Pauli matrices satisfying

$$[\sigma_i^1(x), \sigma_j^2(y)] = 2\mathrm{i}\delta_{i,j}\delta_{x,y}\sigma_i^3(x) \qquad x, y \in \mathbb{Z}^\nu \tag{2}$$

and its cyclic permutations of the components $(1, 2, 3)$. Let

$$\sigma_i^\pm(x) = \frac{\sigma_i^1(x) \pm \mathrm{i}\sigma_i^2(x)}{2}$$

and

$$S^\pm(x) = \sum_{i=-S}^{S} \sigma_i^\pm(x).$$

The Heisenberg model Hamiltonian on Λ is given by

$$H_\Lambda = -\frac{1}{2}\sum_{x,y\in\Lambda}\{J(x,y)[S^1(x)S^1(y) + S^2(x)S^2(y)] + J_3(x,y)S^3(x)S^3(y)\} + h\sum_{x\in\Lambda} S^3(x). \tag{3}$$

We assume finite range, translation invariant interactions, i.e. $J(x, y) = J(|x-y|)$, $J_3(x, y) = J_3(|x-y|)$ and $J(z) = J_3(z) = 0$ for z large enough, we assume also, without loss of generality, $J(x, x) = J_3(x, x) = 0$, which simply amounts to a redefinition of the interaction constants.

We remark that the representation of the spin variables is special in the sense that the $S^\mu(x)$ (1), i.e. per lattice point x, are permutation invariant for arbitrary permutations of the spin index $i = -S, -S+1, \ldots, S$. The Hamiltonian on the other hand is not permutation invariant under permutations of the lattice indices $x \in \mathbb{Z}^\nu$.

As Bloch, Holstein–Primakoff, Dyson [1–5], we are also interested in the S tending to infinity limit. In order to keep the model (3) thermodynamically stable, one has to rescale it by the factor $2S+1$: $H_\Lambda \to H_\Lambda^S = (2S+1)^{-1} H_\Lambda$. Applying a rescaled magnetic field $h \to (2S+1)h$, we get

$$H_\Lambda^S = -\frac{1}{2(2S+1)} \sum_{x,y\in\Lambda} \{J(x,y)[S^1(x)S^1(y) + S^2(x)S^2(y)] + J_3(x,y)S^3(x)S^3(y)\}$$
$$+h \sum_{x\in\Lambda} S^3(x).$$

Or, rewritten, using $J(x,y) = J(y,x)$ and $J(x,x) = 0$:

$$\begin{aligned} H_\Lambda^S &= -\frac{1}{2(2S+1)} \sum_{x,y\in\Lambda} \{2J(x,y)[S^+(x)S^-(y) + S^-(x)S^+(y)] + J_3(x,y)S^3(x)S^3(y)\} \\ &\quad +h \sum_{x\in\Lambda} S^3(x) \\ &= -\frac{1}{2(2S+1)} \sum_{x,y\in\Lambda} \{4J(x,y)S^+(x)S^-(y) + J_3(x,y)S^3(x)S^3(y)\} \\ &\quad +h \sum_{x\in\Lambda} S^3(x). \end{aligned} \tag{4}$$

We are interested in the equilibrium states of this model in the $S \to \infty$ limit, and in the thermodynamic limit $\Lambda \to \mathbb{Z}^\nu$. For each finite volume Λ, H_Λ^S is permutation invariant for one-site observables $\sigma_i^\sharp(x)$ in spin space. Let ω be an equilibrium state, then per lattice site $x \in \Lambda$, the state ω is again permutation invariant. In order to be able to give a definite mathematical meaning to limits of the type, for one-site $x \in \Lambda$,

$$\lim_{S\to\infty} \frac{S^\sharp(x)}{\sqrt{2S+1}}$$

in the sense of non-commutative central limits, the state ω has to satisfy a cluster property

$$\lim_{i\to\infty} \omega(A_i(x)B_j(x)) = \omega(A)\omega(B)$$

where $A(x)$ and $B(x)$ are products of the Pauli matrices (see e.g. [9]). However, all clustering permutation invariant states are product states. Without any restriction of generality, because of the linearity of the equilibrium condition in the state, we can restrict ourself to these product states in spin space. This means that the spin limit is taken with a fixed value of the average spin. This implies that ω, the infinite volume, infinite S limit equilibrium state at inverse temperature β of H_Λ^S, is a product state in spin space:

$$\omega(A_i B_j) = \omega(A)\omega(B)$$

for $i \neq j$ and A, B one-site observables in spin space, i.e. observables on $\otimes_{x\in\Lambda}(\mathbb{C}^2)_x$.

First we remark that for this product state in spin space, one has that in the ω-weak topology [9]:

$$weak - \lim_{S\to\infty} \left(\frac{1}{2S+1} \sum_{i=-S}^{S} \sigma_i^\sharp(x) \right) = \omega(\sigma^\sharp(x)) = \omega(\sigma^\sharp).$$

The last equality follows from space translation invariance, which is a consequence of the space translation invariance of (3).

For $k \in \Lambda^* = \{k = \frac{2\pi}{L} n; n \in \mathbb{Z}^\nu\}$, $|\Lambda| = L^\nu$, let:

$$\sigma_i^\pm(k) = \frac{1}{|\Lambda|^{1/2}} \sum_{x\in\Lambda} (\sigma_i^\pm(x) - \omega(\sigma_i^\pm(x)))\mathrm{e}^{\pm \mathrm{i}k\cdot x}$$

$$\sigma_i^3(k) = \frac{1}{|\Lambda|^{1/2}} \sum_{x\in\Lambda} (\sigma_i^3(x) - \omega(\sigma_i^3(x)))\mathrm{e}^{\mathrm{i}k\cdot x}$$

and

$$\tilde{\sigma}_i^\sharp(k) = \sigma_i^\sharp(k) + |\Lambda|^{1/2}\omega(\sigma^\sharp)\delta_{k,0}.$$

Then

$$F_S^\sharp(k) = \frac{1}{(2S+1)^{1/2}} \sum_{i=-S}^{S} \sigma_i^\sharp(k) \tag{5}$$

are fluctuation operators in spin space and in volume space. The infinite S limit of these operators is known to exist due to the product state character of ω [9], i.e. the following limits exist: for all $\lambda \in \mathbb{R}$,

$$\lim_{S\to\infty} \omega(\exp\{\mathrm{i}\lambda[F_S^\sharp(k) + F_S^\sharp(k)^*]\})$$

and

$$\lim_{S\to\infty} \omega(\exp\{\lambda[F_S^\sharp(k) - F_S^\sharp(k)^*]\}).$$

The limit operators $\lim_{S\to\infty} F_S^\sharp(k)$ are denoted $F^\sharp(k)$, they still depend on the volume Λ. It is straightforward to check that these limits satisfy the canonical commutation relations.

Before taking the infinite S-limit, the operators

$$\mathcal{F}_S^\sharp(k) = \frac{1}{(2S+1)^{1/2}} \sum_{i=-S}^{S} \tilde{\sigma}_i^\sharp(k) \tag{6}$$

can be used to rewrite H_Λ^S (4), after Fourier transformation, in the form

$$H_\Lambda^S = -\sum_{k\in\Lambda^*} \{2J(k)\mathcal{F}_S^+(k)\mathcal{F}_S^-(k) + \tfrac{1}{2}J_3(k)\mathcal{F}_S^3(k)\mathcal{F}_S^3(-k)\} + h(|\Lambda|(2S+1))^{1/2}\mathcal{F}_S^3(0)$$

where $J(k) = \sum_z J(z,0)\mathrm{e}^{-\mathrm{i}k\cdot z}$, $J_3(k) = \sum_z J_3(z,0)\mathrm{e}^{-\mathrm{i}k\cdot z}$.

The following lemma characterizes the rotational invariance around the third axis, if there is enough ferromagnetism.

Lemma 1. *For h sufficiently large, i.e. for*

$$h > \sum_{x\in\Lambda} [J_3(x,0) - J(x,0)] > 0$$

any equilibrium state ω satisfies in the $\Lambda \to \mathbb{Z}^\nu$, $S \to \infty$ limit:

$$\omega(\sigma^\pm) = 0.$$

Proof. Compute

$$\left[H_\Lambda^S, \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma_i^-(x)\right] = \frac{-1}{(2S+1)|\Lambda|} \sum_{j=-S}^{S} \sum_{x,y} \{2J(x,y)\sigma_i^3(x)\sigma_j^-(y)$$

$$-J_3(x,y)[\sigma_i^-(x)\sigma_j^3(y) + \sigma_j^3(y)\sigma_i^-(x)]\} - \frac{2h}{|\Lambda|}\sum_x \sigma_i^-(x).$$

Then the time invariance of ω implies

$$0 = \lim_{|\Lambda|\to\infty} \lim_{S\to\infty} \omega\left(\left[H_\Lambda^S, \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \sigma_i^-(x)\right]\right) = -\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{x,y} \{2J(x,y)\omega(\sigma^3)\omega(\sigma^-)$$
$$-2J_3(x,y)\omega(\sigma^3)\omega(\sigma^-)\} - 2h\omega(\sigma^-)$$

or

$$\omega(\sigma^-)\left\{\omega(\sigma^3)\sum_z [J_3(z,0) - J(z,0)] - h\right\} = 0.$$

Since $-1 \leqslant \omega(\sigma^3) \leqslant 1$, taking $h > \sum_z [J_3(z,0) - J(z,0)] > 0$ ensures

$$\omega(\sigma^3)\sum_z [J_3(z,0) - J(z,0)] - h < 0.$$

Hence

$$\omega(\sigma^-) = 0.$$

□

This result also means that the operators $F_S^\pm$ and $\mathcal{F}_S^\pm$, as defined in (5) and (6) coincide:

$$\mathcal{F}_S^\pm(k) = F_S^\pm(k)$$

and hence also in the infinite S-limit:

$$\mathcal{F}^\pm(k) = F^\pm(k).$$

We compute the commutators for finite S:

$$[\mathcal{F}_S^+(k), \mathcal{F}_S^-(q)] = \frac{1}{|\Lambda|(2S+1)} \sum_{i=-S}^{S} \sum_x [\sigma_i^+(x), \sigma_i^-(x)] \mathrm{e}^{\mathrm{i}(k-q)\cdot x}$$
$$= \frac{1}{|\Lambda|^{1/2}(2S+1)^{1/2}} \mathcal{F}_S^3(k-q)$$

and:

$$[\mathcal{F}_S^3(k), \mathcal{F}_S^\pm(q)] = \frac{1}{|\Lambda|(2S+1)} \sum_{i=-S}^{S} \sum_x [\sigma_i^3(x), \sigma_i^\pm(x)] \mathrm{e}^{\mathrm{i}(k\pm q)\cdot x}$$
$$= \pm\frac{2}{|\Lambda|^{1/2}(2S+1)^{1/2}} \mathcal{F}_S^\pm(q \pm k).$$

Therefore the limits satisfy the boson commutation relations:

$$[\mathcal{F}^+(k), \mathcal{F}^-(q)] = \lim_{S\to\infty} [\mathcal{F}_S^+(k), \mathcal{F}_S^-(q)] = \omega(\sigma^3)\delta_{k,q} \tag{7}$$

and

$$[\mathcal{F}^3(k), \mathcal{F}^\pm(q)] = \lim_{S\to\infty} [\mathcal{F}_S^3(k), \mathcal{F}_S^\pm(q)] = \pm 2\omega(\sigma^\pm)\delta_{k,q} = 0 \tag{8}$$

on the basis of lemma 1.

The $\mathcal{F}^\pm(k)$ are the above rigorously defined magnon creation and annihilation operators. Their existence and explicit properties are established as a straightforward application of the non-commutative central limit theorems in [9], and the condition of ferromagnetism in lemma 1.

Now we proceed by determining the equilibrium state ω completely. We use the definition of equilibrium state by means of correlation inequalities [11], i.e. for temperatures $T > 0$

($\beta = \frac{1}{kT} < \infty$), a state ω is an equilibrium state if and only if it satisfies the energy–entropy balance inequalities:

$$\lim_{\Lambda\to\infty}\lim_{S\to\infty}\beta\omega(X^*[H_\Lambda^S, X]) \geqslant \omega(X^*X)\ln\frac{\omega(X^*X)}{\omega(XX^*)} \tag{9}$$

for all local observables X. We now prove the following theorem.

Theorem 1. *In the ferromagnetic region (see lemma 1), in the infinite S-limit, the equilibrium state ω of the Heisenberg model is a quasi-free state on the fluctuation operators algebra, generated by the $\{\mathcal{F}^\sharp(k), k \in [0, 2\pi]^\nu\}$, completely determined by the two-point function*

$$\omega(\mathcal{F}^+(q)\mathcal{F}^-(q)) = \frac{-\omega(\sigma^3)}{e^{2\beta[-\omega(\sigma^3)(J_3(0)-J(q))+h]}-1}. \tag{10}$$

Proof. First take $X = \mathcal{F}_S^-(q)$, and compute

$$\begin{aligned}
[H_\Lambda^S, \mathcal{F}_S^-(q)] &= -\sum_k 2J(k)[\mathcal{F}_S^+(k), \mathcal{F}_S^-(q)]\mathcal{F}_S^-(k)\\
&\quad -\sum_k \tfrac{1}{2}J_3(k)([\mathcal{F}_S^3(k), \mathcal{F}_S^-(q)]\mathcal{F}_S^3(-k) + \mathcal{F}_S^3(k)[\mathcal{F}_S^3(-k), \mathcal{F}_S^-(q)])\\
&\quad +h|\Lambda|^{1/2}(2S+1)^{1/2}[\mathcal{F}_S^3(0), \mathcal{F}_S^-(q)]\\
&= -\frac{2}{|\Lambda|^{1/2}(2S+1)^{1/2}}\sum_k J(k)\mathcal{F}_S^3(k-q)\mathcal{F}_S^-(k)\\
&\quad +\frac{1}{|\Lambda|^{1/2}(2S+1)^{1/2}}\sum_k J_3(k)(\mathcal{F}_S^-(q-k)\mathcal{F}_S^3(-k) + \mathcal{F}_S^3(k)\mathcal{F}_S^-(q+k))\\
&\quad -2h\mathcal{F}_S^-(q).
\end{aligned}$$

Then:

$$\begin{aligned}
&\beta\omega(\mathcal{F}_S^+(q)[H_\Lambda^S, \mathcal{F}_S^-(q)])\\
&\quad = -\frac{2\beta}{|\Lambda|^{1/2}(2S+1)^{1/2}}\sum_k J(k)\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^3(k-q)\mathcal{F}_S^-(k))\\
&\quad\quad +\frac{\beta}{|\Lambda|^{1/2}(2S+1)^{1/2}}\sum_k J_3(k)\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^-(q-k)\mathcal{F}_S^3(-k))\\
&\quad\quad +\frac{\beta}{|\Lambda|^{1/2}(2S+1)^{1/2}}\sum_k J_3(k)\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^3(k)\mathcal{F}_S^-(q+k))\\
&\quad\quad -2\beta h\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^-(q)).
\end{aligned}$$

Use that for A, B one-site observables in spin space such that $\omega(A) = 0$ (see [9]):

$$\lim_{S\to\infty}\frac{1}{(2S+1)^{1/2}}\omega(\mathcal{F}_S(A)^*\mathcal{F}_S(B)\mathcal{F}_S(A)) = \omega(A^*A)\omega(B)$$

to calculate

$$\begin{aligned}
&\lim_{S\to\infty}\beta\omega(\mathcal{F}_S^+(q)[H_\Lambda^S, \mathcal{F}_S^-(q)])\\
&\quad = -\frac{2\beta}{|\Lambda|^{1/2}}\sum_k J(k)\lim_{S\to\infty}\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^-(k))\omega(\sigma^3)\delta_{k,q}|\Lambda|^{1/2}\\
&\quad\quad +\frac{\beta}{|\Lambda|^{1/2}}\sum_k J_3(k)\lim_{S\to\infty}\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^-(q-k))\omega(\sigma^3)\delta_{k,0}|\Lambda|^{1/2}
\end{aligned}$$

$$+\frac{\beta}{|\Lambda|^{1/2}}\sum_k J_3(k)\lim_{S\to\infty}\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^-(q+k))\omega(\sigma^3)\delta_{k,0}|\Lambda|^{1/2}$$
$$-2\beta h\lim_{S\to\infty}\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^-(q))$$

and

$$\lim_{S\to\infty}\omega(\mathcal{F}_S^+(q)\mathcal{F}_S^-(q)) = \omega(\mathcal{F}^+(q)\mathcal{F}^-(q)).$$

Then

$$\lim_{\Lambda\to\infty}\lim_{S\to\infty}\beta\omega(\mathcal{F}_S^+(q)[H_\Lambda^S,\mathcal{F}_S^-(q)]) = 2\beta\{\omega(\sigma^3)[J_3(0)-J(q)]-h\}\omega(\mathcal{F}^+(q)\mathcal{F}^-(q)).$$

After substitution into the correlation inequality (9), one gets:

$$-2\beta\{-\omega(\sigma^3)[J_3(0)-J(q)]+h\}\geqslant \ln\frac{\omega(\mathcal{F}^+(q)\mathcal{F}^-(q))}{\omega(\mathcal{F}^-(q)\mathcal{F}^+(q))}.$$

Interchanging the role of $\mathcal{F}^+(q)$ and $\mathcal{F}^-(q)$, i.e. take now $X=\mathcal{F}_S^+(q)$ in (9) and repeat the computation above, then:

$$2\beta\{-\omega(\sigma^3)[J_3(0)-J(q)]+h\}\geqslant \ln\frac{\omega(\mathcal{F}^-(q)\mathcal{F}^+(q))}{\omega(\mathcal{F}^+(q)\mathcal{F}^-(q))}.$$

These two inequalities, combined with the commutation relation (7) yield

$$\ln\frac{\omega(\mathcal{F}^+(q)\mathcal{F}^-(q))-\omega(\sigma^3)}{\omega(\mathcal{F}^+(q)\mathcal{F}^-(q))} = 2\beta\{-\omega(\sigma^3)[J_3(0)-J(q)]+h\}$$

or the expected two-point function

$$\omega(\mathcal{F}^+(q)\mathcal{F}^-(q)) = \frac{-\omega(\sigma^3)}{\mathrm{e}^{2\beta\{-\omega(\sigma^3)[J_3(0)-J(q)]+h\}}-1}.$$

Finally if one takes for X higher-order monomials in the $\mathcal{F}_S^\sharp(q)$, one derives readily also from (9), that the higher-order point correlation functions are sums of products of this two-point function, proving that the state ω is quasi-free. As this amounts to a straightforward computation, we leave it as an exercise for the reader. □

The basic two-point function (10) still contains the magnetization $\omega(\sigma^3)$. Using (2) and (6), one gets a self-consistency equation for the magnetization

$$\frac{1}{|\Lambda|}\sum_{q\in\Lambda^*}\omega(\mathcal{F}^+(q)\mathcal{F}^-(q)) = \frac{1+\omega(\sigma^3)}{2}. \tag{11}$$

Let

$$D(x,y) = \lambda(x)\delta_{x,y} - J(x,y) \tag{12}$$

where

$$\lambda(x) = \sum_y J_3(x,y)$$

and suppose that we limit ourself to the *ferromagnetic situation* (e.g. $J_3(x,y)\geqslant |J(x,y)|$, see [8]), expressed by the condition that the matrix D (12) is positive definite:

$$D(x,y)\geqslant 0.$$

This implies

$$D(q) = \sum_z D(z,0)\mathrm{e}^{-\mathrm{i}q\cdot z} = J_3(0)-J(q)\leqslant \sum_z D(z,0) = D(0).$$

Hence

$$h > \sum_z [J_3(z,0) - J(z,0)] = D(0) \geqslant D(q)$$

and

$$h - \omega(\sigma^3)[J_3(0) - J(q)] = h - \omega(\sigma^3)D(q) \geqslant h - D(q) \geqslant h - D(0) > 0.$$

From this, and from (10), it follows that

$$0 \leqslant \omega(\mathcal{F}^+(q)\mathcal{F}^-(q)) = \frac{-\omega(\sigma^3)}{\mathrm{e}^{2\beta\{h-\omega(\sigma^3)D(q)\}} - 1} \leqslant \frac{1}{\mathrm{e}^{2\beta h} - 1}.$$

The first inequality yields $-1 \leqslant \omega(\sigma^3) \leqslant 0$.

Using (11), one gets

$$\omega(\sigma^3) \leqslant -1 + \frac{2}{\mathrm{e}^{2\beta h} - 1} \tag{13}$$

$$\leqslant -1 + \frac{2}{\mathrm{e}^{2\beta D(0)} - 1} \simeq -1 + 2\mathrm{e}^{-2\beta D(0)} \tag{14}$$

establishing a bound on the magnetization for low temperatures as a function of the interaction constants. The bound measures the deviation of the magnetization from its ground state value, equal to -1, for small temperatures.

Remark that for $\omega(\sigma^3) = -1$, the ground state ($\beta \to \infty$) value of the ferromagnetic system, the magnon creation and annihilation operators form a bosonic pair satisfying the canonical commutation relations

$$[\mathcal{F}^-(q), \mathcal{F}^+(q)] = 1$$

and that the ground state ω is a magnon Fock state

$$\omega(\mathcal{F}^+(q)\mathcal{F}^-(q)) = \lim_{\beta\to\infty} \frac{1}{\mathrm{e}^{2\beta[D(q)+h]} - 1} = 0.$$

In general, for all temperatures, the magnetization $\omega(\sigma^3)$ plays the role of the quantization parameter (a Planck constant) (see (7)) for the field of magnons. All quantum character of the magnons vanishes if one chooses the magnetic field h and/or the interaction constants ($D(q)$) and/or the temperature such that the magnetization vanishes.

Concerning the magnetization fluctuation operators $F_S^3(q)$, the ferromagnetic conditions ($h - D(0) > 0$, $D \geqslant 0$) are such that its infinite S-limit $F^3(q)$ commutes with all other magnon observables (see (8)). They become classical observables, and disappear completely from the action of the system Hamiltonian. It does not mean that there are no magnetization fluctuations.

The original system with Hamiltonian H_Λ^S in terms of the fluctuation observables $\{\mathcal{F}_S^\pm(q), F_S^3(q)\}_{q\in\Lambda^*}$, becomes in the infinite S-limit a system of non-interacting magnons with the Hamiltonian

$$\mathcal{H}_\Lambda = \sum_{q\in\Lambda^*} \epsilon(q)\mathcal{F}^+(q)\mathcal{F}^-(q) \tag{15}$$

where the magnon fluctuation creation and annihilation operators satisfy

$$[\mathcal{F}^-(q), \mathcal{F}^+(q')] = -\omega(\sigma^3)\delta_{q,q'}$$

and where the spectrum is given by

$$\epsilon(q) = 2\left(J_3(0) - J(q) + \frac{h}{-\omega(\sigma^3)}\right).$$

Note that if $\min(h, D(0)) > 0$, then already $\epsilon(q = 0) > 0$, i.e. there is no condensation of magnons in the zero mode $q = 0$. In particular, this is the case under our assumptions.

By inverse Fourier transform, i.e.

$$\mathcal{F}^{\pm}(x) = \frac{1}{|\Lambda|^{1/2}} \sum_{q \in \Lambda^*} \mathcal{F}^{\pm}(q) \mathrm{e}^{\mp \mathrm{i} q \cdot x}$$

(15) can be written in configuration space:

$$\mathcal{H}_\Lambda = 2 \sum_{x,y} D(x, y) \mathcal{F}^+(x) \mathcal{F}^-(y) + \frac{2h}{-\omega(\sigma^3)} \sum_x \mathcal{F}^+(x) \mathcal{F}^-(x) \tag{16}$$

with

$$[\mathcal{F}^-(x), \mathcal{F}^+(y)] = -\omega(\sigma^3) \delta_{x,y} \tag{17}$$

and D is as defined above. Written this way, it is clear that $\mathcal{H}_\Lambda$ is nothing but the 'quadratic boson Hamiltonian' of [8].

Finally we look for the dynamics of the magnon excitation number operator $\mathcal{F}^+(x)\mathcal{F}^-(x)$. On the basis of (11) the expectation value of this operator is related to the magnetization. Therefore, the time evolution of this number operator is related to the time evolution of the magnetic moment in a ferromagnet.

The equation of motion for $\mathcal{F}^+(x)\mathcal{F}^-(x)$ is given by

$$\begin{aligned}\frac{\partial}{\partial t} \mathcal{F}^+(x) \mathcal{F}^-(x) &= \mathrm{i}[\mathcal{H}_\Lambda, \mathcal{F}^+(x) \mathcal{F}^-(x)] \\ &= -2\omega(\sigma^3) \sum_y J(x, y)(\mathcal{F}^+(y)\mathcal{F}^-(x) - \mathcal{F}^-(y)\mathcal{F}^+(x)).\end{aligned} \tag{18}$$

Indeed, using (17) and $J(x, x) = 0$ straightforwardly yields the result.

This dynamical equation (18) can be compared with the macroscopic equation of motion for the magnetic moment, first derived in [12] (see also [13,14]). A recent rigorous derivation of this equation for zero temperature is given in [15], applying a hydrodynamical limit or a Lebowitz–Penrose approximation [16]. We remark that equation (18), however, is valid for all temperature equilibrium states.

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