Commun. Math. Phys. 216, 461 – 490 (2001)

Communications in
Mathematical Physics
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Goldstone Boson Normal Coordinates

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Received: 2 February 2000 / Accepted: 11 September 2000

Abstract: The phenomenon of spontaneous symmetry breaking is well known. It is known to be accompanied with the appearance of the "Goldstone boson". In this paper we construct the canonical coordinates of the Goldstone boson, for quantum spin systems with short range as well as long range interactions.

1. Introduction

As is well known, spontaneous symmetry breakdown (SSB) is one of the basic phenomena accompanying collective phenomena, such as phase transitions in statistical mechanics, or ground state excitations in field theory. SSB is a representative tool for the analysis of many phenomena in modern physics. The study of SSB goes back to the Goldstone Theorem [1], which was the subject of much analysis. This theorem refers usually to the ground state property that for short range interacting systems, SSB implies the absence of an energy gap in the excitation spectrum [2,3].

In this paper we concentrate on the non-relativistic Goldstone Theorem, and we mean by this spontaneous symmetry breaking of a continuous symmetry group in condensed matter homogeneous many particles systems, with short range as well as long range interactions.

There are many different situations to consider. For short range interactions, it is typical that SSB yields a dynamics which remains symmetric in the thermodynamic limit. At temperature $T = 0$, one has as main characteristics the absence of an energy gap. However for equilibrium states ($T > 0$), SSB is better characterized by bad clustering properties [4,5].

For long range interactions, it is typical that SSB breaks also the symmetry of the dynamics. This situation has been studied extensively in the literature. In physics the phenomenon is known as the occurrence of oscillations with energy spectrum taking a

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finite value $\epsilon(k \to 0) \neq 0$. Different approximation methods, typical here is the random phase approximation, yield the computation of these frequencies. For mean field models, such as the BCS-model [6], the Overhauser model [7], a spin density wave model [8], the anharmonic crystal model [9], and for the jellium model [10], one is able to give the rigorous mathematical status of these frequencies as elements of the spectrum of typical fluctuation operators [11,12]. The typical operators entering in the discussion are the generator of the broken symmetry and the order parameter operator. In a physical language they are the charge density and current density operators. It is proved that their fluctuation operators form a quantum canonical pair, which decouples from the other degrees of freedom of the system. As fluctuation operators are collective operators, they describe the collective mode accompanying the SSB phenomenon. Hence for long range interacting systems, we realised mathematically rigorously in these models, the so-called Anderson theorem [13,14] of "restoration of symmetry", stating that there exists a spectrum of collective modes $\epsilon(k \to 0) \neq 0$ and that the mode in the limit $k \to 0$ is the operator which connects the set of degenerate temperature states, i.e. "rotates" one ergodic state into another. We conjecture that our results of [6–10] can be proved for general long range two-body interacting systems as a universal theorem.

However Anderson did formulate his theorem in the context of the Goldstone theorem for short range interacting systems, i.e. in the case $\epsilon(k \to 0) = 0$ of absence of an energy gap in the ground state. Of course one knows that there is no one-to-one relation between long range interactions and the presence of an energy gap for symmetry breaking systems (see e.g. [9]). The imperfect Bose gas and the weakly interacting Bose gas are examples of long range interacting systems showing SSB, but without energy gap. In [15] we realise for these boson models the above described programme of construction of the collective modes operators of condensate density and condensate current, as normal modes dynamically independent from the other degrees of freedom of the system. We consider the whole temperature range, the ground state included. In particular the ground state situation is interesting, because it yields a non-trivial quantum mechanical canonical pair of conjugate operators, giving an explicit representation of the field variables of the so-called Goldstone boson.

In this paper we are able to present the analogous proof for general interacting quantum lattice systems, and hence give *a model independent construction*. We construct the fluctuation operators of the generator of a broken symmetry and of the order parameter and prove that they form a canonical pair. We prove that this pair is dynamically independent from the other degrees of freedom of the system.

In the case of long range interactions, we prove that the appearance of a plasmon fequency is a natural phenomenon corresponding to the spectrum of the above mentioned canonical pair. Moreover these fluctuation operators are normal. Our main contribution here is *the construction of a canonical order parameter*. Usually there are many order parameter operators. Therefore the identification of the right one for the purpose is important.

For short range interactions in the ground state, we find again the phenomenon of squeezing of the fluctuation operator of the generator of the broken symmetry. In the literature this is sometimes referred to the statement that in case of SSB, the broken symmetry behaves like an approximate symmetry. The amount of squeezing is inversely related to the anormality of the fluctuation operator of the order parameter, which itself is directly related to the degree of off-diagonal long range order. Using an appropriate volume scaling, which is determined by the long wavelength behaviour of the spectrum, we arrive at the construction of the Goldstone boson normal coordinates. We consider this result as a formal step forward, beyond the known analysis of the Goldstone phenomenon. We repeat that our construction is solely determined by the long wavelength behaviour of the microscopic energy spectrum of the system.

Finally, we want to throw the attention of the reader to the direct open questions which should keep our attention. There is first of all the problem of SSB of more dimensional symmetries. One should expect a more dimensional Goldstone boson. There is also the problem of SSB of non-commutative symmetry groups. An insight in this situation would certainly contribute to information on the situation of SSB in gauge theories in relativistic field theory.

2. Canonical Coordinates

2.1. Introduction. In [11, 12] a dynamical system of macroscopic quantum fluctuations is constructed for sufficiently clustering states. We repeat the main results in order to fix the notation and refer to the original papers for more details and proofs. The main issue of this section is the construction of creation and annihilation operators for this system of macroscopic fluctuation observables. We start by formulating the systems and the technical settings.

With each $x \in \mathbb{Z}^\nu$ we associate the algebra $\mathcal{A}_x$, a copy of the matrix algebra M_N of $N \times N$ matrices. For each $\Lambda \subset \mathbb{Z}^\nu$, consider the tensor product $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_x$. The algebra of all local observables is

$$\mathcal{A}_L = \bigcup_{\Lambda\subset\mathbb{Z}^\nu} \mathcal{A}_\Lambda .$$

The norm closure $\mathcal{A}$ of $\mathcal{A}_L$ is again a C^*-algebra

$$\mathcal{A} = \overline{\mathcal{A}_L} = \overline{\bigcup_{\Lambda\subset\mathbb{Z}^\nu} \mathcal{A}_\Lambda},$$

and is considered the algebra of quasi-local observables of our system.

The group $\mathbb{Z}^\nu$ of space translations of the lattice acts as a group of *-automorphisms on $\mathcal{A}$ by:

$$\tau_x : A \in \mathcal{A}_\Lambda \to \tau_x(A) \in \mathcal{A}_{\Lambda+x} \ , \quad x \in \mathbb{Z}^\nu .$$

The dynamics of our system is determined in the usual way by the local Hamiltonians

$$H_\Lambda = \sum_{X\subseteq\Lambda} \Phi(X), \quad \Lambda \subset \mathbb{Z}^\nu$$

with self adjoint $\Phi(X) \in \mathcal{A}_X$ for all $X \subset \mathbb{Z}^\nu$. The interaction Φ is supposed to be translation invariant:

$$\tau_x \Phi(X) = \Phi(X + x).$$

For each $\Lambda \subset \mathbb{Z}^\nu$, the local dynamics α_t^Λ is given by

$$\alpha_t^\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda,$$
$$\alpha_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad A \in \mathcal{A}_\Lambda .$$

If there exists $\lambda > 0$ such that

$$\|\Phi\|_\lambda \equiv \sum_{0\in X} |X| N^{2|X|} e^{\lambda d(X)} \|\Phi(X)\| < \infty, \tag{1}$$

with $d(X) = \sup_{x,y\in X} |x-y|$ the diameter of the set X and $|X|$ the number of elements in X, then the global dynamics α_t is well defined as the norm limit of the local dynamics α_t^Λ [16].

The state ω is an (α_t, β)-KMS state which is supposed to have good spatial clustering expressed by

$$\sum_{x\in\mathbb{Z}^\nu} \alpha_\omega(|x|) < \infty, \tag{2}$$

with α_ω the following clustering function:

$$\alpha_\omega(d) = \sup_{\Lambda,\Lambda'} \sup_{A\in\mathcal{A}_\Lambda, B\in\mathcal{A}_{\Lambda'}} \left\{ \frac{1}{\|A\|\|B\|} |\omega(AB) - \omega(A)\omega(B)| \;\middle|\; d \le d(\Lambda, \Lambda') \right\}. \tag{3}$$

Through the GNS construction, ω defines the Gelfand triple $(\mathcal{H}, \pi, \Omega)$, where $\mathcal{H}$ is a Hilbert space, π a *-representation of $\mathcal{A}$ as bounded operators on $\mathcal{H}$ and Ω a cyclic vector of $\mathcal{H}$ such that

$$\omega(A) = (\Omega, \pi(A)\Omega).$$

2.2. Normal fluctuations. Denote by Λ_n the cube centered around the origin with edges of length $2n+1$. For any $A \in \mathcal{A}$, the local fluctuation $F_n(A)$ of A in the state ω is given by

$$F_n(A) = \frac{1}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n} \big(\tau_x A - \omega(A)\big).$$

In [12] it is proved that under the condition (2), the central limits exist: for all $A, B \in \mathcal{A}_{L,sa}$ (self-adjoint elements of $\mathcal{A}_L$)

$$\begin{aligned}\lim_{n\to\infty} \omega\big(e^{iF_n(A)} e^{iF_n(B)}\big) &= \lim_{n\to\infty} \omega\big(e^{iF_n(A+B)}\big) e^{-\frac{1}{2}\omega([F_n(A),F_n(B)])} \\ &= \exp\left\{-\frac{1}{2} s_\omega\,(A+B, A+B) - \frac{i}{2}\sigma_\omega(A,B)\right\},\end{aligned}$$

where

$$\begin{aligned} s_\omega(A,B) &= \lim_{n\to\infty} \text{ Re } \omega\big(F_n(A)^* F_n(B)\big) = \text{Re } \sum_{x\in\mathbb{Z}^\nu} \big(\omega(A^*\tau_x B) - \omega(A^*)\omega(B)\big), \\ \sigma_\omega(A,B) &= \lim_{n\to\infty} 2 \text{ Im } \omega\big(F_n(A)^* F_n(B)\big) = -i \sum_x \omega\big([A, \tau_x B]\big).\end{aligned}$$

Now we are able to introduce the algebra of normal fluctuations of the system $(\mathcal{A}, \mathcal{A}_L, \omega)$. Consider the symplectic space $(\mathcal{A}_{L,sa}, \sigma_\omega)$. Denote by $W(\mathcal{A}_{L,sa}, \sigma_\omega)$ the

CCR-algebra generated by the Weyl operators $\{W(A)|A \in \mathcal{A}_{L,sa}\}$, satisfying the product rule

$$W(A)W(B) = W(A+B)e^{-\frac{i}{2}\sigma_\omega(A,B)}.$$

The central limit theorem fixes a representation of this CCR-algebra in the following way. For each $A \in \mathcal{A}_{L,sa}$ the limits $\lim_{n\to\infty} \omega\big(e^{iF_n(A)}\big)$ define a quasi-free state $\tilde{\omega}$ of the CCR-algebra $W(\mathcal{A}_{L,sa}, \sigma_\omega)$ by

$$\tilde{\omega}\big(W(A)\big) = e^{-\frac{1}{2}s_\omega(A,A)}.$$

Moreover if γ is a *-automorphism of $\mathcal{A}$ leaving $\mathcal{A}_L$ invariant, commuting with the space translations and leaving the state ω invariant, then $\tilde{\gamma}$ given by

$$\tilde{\gamma}(W(A)) = W(\gamma(A)) \tag{4}$$

defines a quasi-free *-automorphism of $W(\mathcal{A}_{L,sa}, \sigma_\omega)$.

The quasi-free state $\tilde{\omega}$ induces a GNS-triplet $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\Omega})$ and yields a von Neumann algebra

$$\tilde{\mathcal{M}} = \tilde{\pi}\big(W(\mathcal{A}_{L,sa}, \sigma_\omega)\big)''.$$

This algebra will be called the *algebra of normal (macroscopic) fluctuations*.

By the fact that the representation $\tilde{\pi}$ is regular, we can define boson fields $F_0(A)$ given by $\tilde{\pi}(W(A)) = e^{iF_0(A)}$, and satisfying

$$[F_0(A), F_0(B)] = i\sigma_\omega(A,B).$$

Through the relation

$$\lim_{n\to\infty} \omega\big(e^{iF_n(A)}\big) = \tilde{\omega}\big(e^{iF_0(A)}\big),$$

we are able to identify the macroscopic fluctuations of the system $(\mathcal{A}, \omega)$ with the boson field $F_0(\cdot)$:

$$\lim_{n\to\infty} F_n(A) = F_0(A).$$

Let $(\mathcal{H}, \pi, \Omega)$ be the GNS-triplet induced by the state ω and consider the sesquilinear form $\langle\cdot,\cdot\rangle_0$ on $\mathcal{H}$ with domain $\pi(\mathcal{A}_L)\Omega$ which we simply denote by $\mathcal{A}_L$:

$$\langle A, B\rangle_0 = s_\omega(A,B) + \frac{i}{2}\sigma_\omega(A,B) = \sum_{x\in\mathbb{Z}^\nu} \big(\omega(A^*\tau_x B) - \omega(A^*)\omega(B)\big).$$

We call A and B in $\mathcal{A}_L$ equivalent, denoted $A \equiv_0 B$ if $\langle A-B, A-B\rangle_0 = 0$. The following important result holds:

$$A \equiv_0 B \Leftrightarrow \tilde{\pi}\left(W(A)\right) = \tilde{\pi}\left(W(B)\right). \tag{5}$$

This is the property of *coarse graining*: different micro observables yield the same macroscopic fluctuation operator.

Denote by $[\mathcal{A}_L]$ the equivalence classes of $\mathcal{A}_L$ for the equivalence relation $\equiv_0$. The form $\langle\cdot,\cdot\rangle_0$ is a scalar product on $[\mathcal{A}_L]$. Denote by $\mathcal{K}_\omega$ the Hilbert space obtained as the completion of $[\mathcal{A}_L]$. Clearly s_ω and σ_ω extend continuously to $\mathcal{K}_\omega$. Denote by $\mathcal{K}_\omega^{\text{Re}}$ the real subspace of $\mathcal{K}_\omega$ generated by $[\mathcal{A}_{L,sa}]$. Now one considers the CCR-algebra $W\left(\mathcal{K}_\omega^{\text{Re}}, \sigma_\omega\right)$ in the same representation induced by the state $\tilde{\omega}$, and one has the following equality:

$$\tilde{\mathcal{M}} = \tilde{\pi}\big(W(\mathcal{K}_\omega^{\text{Re}}, \sigma_\omega)\big)''.$$

2.3. Reversible dynamics of fluctuations. Property (4) is not directly applicable with $\gamma = \alpha_t$, because with this choice it is not clear, and generally not true that $\alpha_t \mathcal{A}_L \subseteq \mathcal{A}_L$. Nevertheless, since $\alpha_t F_n(A) = F_n(\alpha_t A)$ one is tempted to define the dynamics $\tilde{\alpha}_t$ of the fluctuations by the formula

$$\tilde{\alpha}_t F_0(A) = F_0(\alpha_t A).$$

The non-trivial point in this formula is that it is unclear whether the central limit of the non-local observable $\alpha_t A$ exists or not. Furthermore if $F_0(\alpha_t A)$ exists it remains to prove that $(\tilde{\alpha}_t)_t$ defines a weakly continuous group of *-automorphisms on the fluctuation algebra $\tilde{\mathcal{M}}$.

In [12] it is shown that if the interaction Φ is of short range, i.e. if Φ satisfies condition (1), then for all $A \in [\mathcal{A}_L]$, one has that for all $t \in \mathbb{R}$, $\alpha_t A \in \mathcal{K}_\omega$ and if $A \in [\mathcal{A}_{L,sa}]$ then $\alpha_t A \in \mathcal{K}_\omega^{\mathrm{Re}}$. $W(\alpha_t A)$ is a well defined element of $\tilde{\mathcal{M}}$ and as

$$W(\alpha_t A) = e^{iF_0(\alpha_t A)}, \ \ A \in [\mathcal{A}_{L,sa}]$$

the fluctuation $F_0(\alpha_t A)$ exists for all $t \in \mathbb{R}$.

The map $U_t : [\mathcal{A}_L] \to \mathcal{K}_\omega, U_t A = \alpha_t A$ is a well defined linear operator on the Hilbert space $(\mathcal{K}_\omega, \langle\cdot,\cdot\rangle_0)$ extending to a unitary operator for all $t \in \mathbb{R}$. The map $t \to U_t$ is a strongly continuous one-parameter group, and for all elements $A \in \mathcal{K}_\omega^{\mathrm{Re}}$ we can define $\tilde{\alpha}_t W(A) = W(U_t A)$. Then $\tilde{\alpha}_t$ extends to a weakly continuous one-parameter group of *-automorphisms of $\tilde{\mathcal{M}}$.

Moreover it is shown that if the microsystem is in an equilibrium state, then also the macro system of fluctuations is in an equilibrium state for the dynamics constructed in the previous theorem, i.e. the notion of equilibrium is preserved under the operation of coarse graining induced by the central limit. In particular, if ω is an α_t-KMS state of $\mathcal{A}$ at $\beta > 0$, then $\tilde{\omega}$ is an $\tilde{\alpha}_t$-KMS state of the von Neumann algebra $\tilde{\mathcal{M}}$ at the same temperature.

2.4. Canonical coordinates. Now we proceed to the explicit construction of creation and annihilation operators of fluctuations in the algebra $\tilde{\mathcal{M}}$. For product states this construction can be found in [17]. Here we work out the construction for the most general system.

From the definition of $\mathcal{K}_\omega^{\mathrm{Re}}$ and $\mathcal{K}_\omega$ we can write

$$\mathcal{K}_\omega = \mathcal{K}_\omega^{\mathrm{Re}} + i\mathcal{K}_\omega^{\mathrm{Re}}.$$

Let * be the operation on $\mathcal{K}_\omega$ defined by

$$A^* = (A_1 + iA_2)^* = A_1 - iA_2, \quad A_1, A_2 \in \mathcal{K}_\omega^{\mathrm{Re}}.$$

Clearly for $X \in \mathcal{A}_L$ one has $[X]^* = [X^*]$ and it follows from the properties of U_t (see above) that

$$(U_t A)^* = U_t A^*$$

for all $A \in \mathcal{K}_\omega$.

Let $\mathcal{D}$ denote the set of infinitely differentiable functions on $\mathbb{R}$ with compact support. $\mathcal{D}$ is dense in $\mathcal{C}_0(\mathbb{R})$, the continuous functions vanishing at ∞, for the supremum norm. If $\hat{f} \in \mathcal{D}$ then the inverse Fourier transform

$$f(z) = \int_{-\infty}^{+\infty} d\lambda \hat{f}(\lambda) e^{i\lambda z}$$

is an entire analytic function. If supp $\hat{f} \in [-R, R]$ then it follows from the theorem of Paley–Wiener [16] that for all $n \in \mathbb{N}$ there exists a constant C_n such that

$$|f(z)| \leq C_n (1 + |z|)^{-n} e^{R|\mathrm{Im} z|}.$$

Let $U_t = e^{it\tilde{h}} = \int e^{it\lambda} d\tilde{E}_\lambda$ be the spectral resolution of the unitary group U_t and for $A \in \mathcal{K}_\omega$, $f \in L^1(\mathbb{R})$ denote

$$A(f) = \int_{-\infty}^{+\infty} dt f(t) U_t A = \int_{-\infty}^{+\infty} \hat{f}(-\lambda) d\tilde{E}_\lambda A = \hat{f}(-\tilde{h}) A.$$

Clearly one has $A(f)^* = A^*(\bar{f})$.

Let W be an open set in $\mathbb{R}$ and let $\tilde{E}_W = \int_W d\tilde{E}_\lambda$ be the spectral projection onto the spectral subspace $\mathcal{K}_W$. It follows from the spectral theory [16,18] that $\mathcal{K}_W$ is generated by the set

$$\{A(f) | A \in \mathcal{K}_\omega,\ f \in \mathcal{D},\ \mathrm{supp}\ \hat{f} \subset W\}.$$

Finally for $A \in \mathcal{K}_\omega$ denote the associated spectral measure by

$$d\tilde{\mu}_A(\lambda) = \langle A, d\tilde{E}_\lambda A \rangle_0$$

and its spectral support Δ_A

$$\Delta_A = \{\lambda \in \mathbb{R} \mid \tilde{\mu}_A([\lambda - \epsilon, \lambda + \epsilon]) > 0\ \forall \epsilon > 0\}. \tag{6}$$

It is easy to see that Δ_A is also given by $\Delta_A = \{\underline{\lambda} \in \mathbb{R} \mid \hat{f}(\lambda) = 0,\ \forall \hat{f} \in \mathcal{D}$ such that $A(f) = 0\}$. From this expression and $\hat{\bar{f}}(\lambda) = \overline{\hat{f}(-\lambda)}$ it follows that $\Delta_{A^*} = -\Delta_A$, and from the same argument one also has

$$\tilde{E}_+ A^* = (\tilde{E}_- A)^*, \tag{7}$$

where $\tilde{E}_+ = \tilde{E}_{(0,+\infty)}$ and $\tilde{E}_- = \tilde{E}_{(-\infty,0)}$ are the projections onto positive, respectively negative energy.

Lemma 1. *Let ω be an (α_t, β)-KMS state on the algebra $\mathcal{A}$. For all $A \in \mathcal{K}_\omega$, $\hat{f} \in \mathcal{D}$*

$$\int \hat{f}(\lambda) d\mu_A(\lambda) = \int \hat{f}(\lambda) e^{\beta\lambda} d\mu_{A^*}(-\lambda).$$

Proof. Follows from the KMS-properties of $\tilde{\omega}$. □

Let $\mathcal{K}^{\mathrm{Re}}_{\omega,0} = \tilde{E}_0 \mathcal{K}^{\mathrm{Re}}_\omega$ and $\mathcal{K}^{\mathrm{Re}}_{\omega,1} = (\tilde{E}_+ + \tilde{E}_-)\mathcal{K}^{\mathrm{Re}}_\omega$. Define the operator J on $\mathcal{K}^{\mathrm{Re}}_{\omega,1}$ by

$$J = i(\tilde{E}_+ - \tilde{E}_-). \tag{8}$$

From (7) one has for all $A \in \mathcal{K}^{\mathrm{Re}}_{\omega,1}$, $(JA)^* = JA^*$ and thus $J\mathcal{K}^{\mathrm{Re}}_{\omega,1} \subseteq \mathcal{K}^{\mathrm{Re}}_{\omega,1}$.

Proposition 2. *The operator J defined above is a complex structure on the symplectic space $(\mathcal{K}^{\mathrm{Re}}_{\omega,1}, \sigma_\omega)$:*

(i) $J^2 = -1$,
(ii) $\sigma_\omega(A, JB) = -\sigma_\omega(JA, B), \quad A, B \in \mathcal{K}^{\mathrm{Re}}_{\omega,1}$,
(iii) $\sigma_\omega(A, JA) > 0, \quad 0 \neq A \in \mathcal{K}^{\mathrm{Re}}_{\omega,1}$.

Proof. From the definition of J and $\sigma_\omega = 2\,\mathrm{Im}\,\langle\cdot,\cdot\rangle_0$, (i) and (ii) are trivially satisfied. Now we prove (iii). Let $\mathcal{E}$ be the set of real functions f such that $\hat{f} \in \mathcal{D}$ and $0 \notin \mathrm{supp}\,\hat{f}$. By the spectral theory, the set generated by $\{A(f)|A \in \mathcal{K}^{\mathrm{Re}}_{\omega,1},\ f \in \mathcal{E}\}$ is dense in $\mathcal{K}^{\mathrm{Re}}_{\omega,1}$.

Take such an element $A(f)$. Using the previous lemma one computes

$$\begin{aligned}\langle \tilde{E}_- A(f), \tilde{E}_- A(f)\rangle_0 &= \int |\hat{f}(\lambda)|^2 \chi_{(-\infty,0)}(\lambda) d\mu_A(\lambda)\\ &= \int |\hat{f}(-\lambda)|^2 e^{-\beta\lambda} \chi_{(0,\infty)}(\lambda) d\mu_A(\lambda)\\ &= \langle \tilde{E}_+ A(f), e^{-\beta\tilde{h}} \tilde{E}_+ A(f)\rangle_0.\end{aligned}$$

Because $\tilde{E}_+, \tilde{E}_-$ are projections and $e^{-\beta\tilde{h}} = \int e^{-\beta\lambda} d\tilde{E}_\lambda$ is bounded on $\tilde{E}_+\mathcal{K}^{\mathrm{Re}}_{\omega,1}$, this relation holds for all $B \in \mathcal{K}^{\mathrm{Re}}_{\omega,1}$. Using this property one has

$$\begin{aligned}\sigma_\omega(A, JA) &= -2i\,\mathrm{Im}\,\langle A, JA\rangle_0 = 2\left(\langle \tilde{E}_+ A, \tilde{E}_+ A\rangle_0 - \langle \tilde{E}_- A, \tilde{E}_- A\rangle_0\right)\\ &= 2\int_0^\infty (1 - e^{-\beta\lambda})\langle A, d\tilde{E}_\lambda A\rangle_0 \geq 0.\end{aligned}$$

The strict inequality holds because the spectral measure $d\tilde{\mu}_A(\lambda)$ is regular and $\tilde{E}_0 A = 0$. □

The existence of a complex structure J yields the existence of creation and annihilation operators

$$a_0^{\pm}(A) = \frac{F_0(A) \mp i F_0(JA)}{\sqrt{2}} \tag{9}$$

for all $A \in \mathcal{K}^{\mathrm{Re}}_{\omega,1}$. They satisfy the property

$$a_0^{\pm}(JA) = \pm i a_0^{\pm}(A).$$

2.5. Normal modes. Consider a given microscopic observable A such that $[A] \in \mathcal{K}^{\mathrm{Re}}_{\omega,1}$, i.e. such that $F_0(A)$ evolves non-trivially under the dynamics $\tilde{\alpha}_t$. For simplicity we will denote $A = [A]$. We will construct the normal modes corresponding to the macroscopic fluctuations of the observable A.

In order to make clear the idea we will first make the simplifying assumption that the spectral measure $d\tilde{\mu}_A(\lambda)$ consists of two δ-peaks, at $\pm\epsilon_A$, with $\epsilon_A > 0$. Afterwards we will show how to extend the construction to more general (absolutely continuous) measures $d\tilde{\mu}_A$. Notice also that the prototype examples of systems with normal fluctuations, i.e. mean field systems, have a discrete energy spectrum and therefore obey the δ-peak assumption (see Sect. 3 for an explicit example).

Lemma 3. *For* $\hat{f} \in \mathcal{D}$ *and* $[A] \in \mathcal{K}^{\mathrm{Re}}_{\omega,1}$,

$$\int \hat{f}(\lambda) d\tilde{\mu}_A(\lambda) = \int_0^\infty \big(\hat{f}(\lambda) + \hat{f}(-\lambda) e^{-\beta\lambda}\big) d\tilde{\mu}_A(\lambda),$$

and for $\hat{f}(\tilde{h})A \in \mathcal{K}^{\mathrm{Re}}_{\omega}$ *(i.e.* $f(t)$ *real),*

$$\tilde{\omega}\Big(F_0\big(\hat{f}(\tilde{h})A\big)^2\Big) = \int |\hat{f}(\lambda)|^2 d\tilde{\mu}_A(\lambda).$$

Proof. This is a simple computation and application of Lemma 1. □

It will turn out to be more natural to work in terms of the following measure: for $\lambda > 0$,

$$dc_A(\lambda) \equiv 2 \frac{1 - e^{-\beta\lambda}}{\lambda} d\tilde{\mu}_A(\lambda),$$

and 0 otherwise, such that by Lemma 3,

$$c_A \equiv \int_0^\infty dc_A(\lambda) = \int_{-\infty}^{+\infty} \frac{1 - e^{-\beta\lambda}}{\lambda} d\tilde{\mu}_A(\lambda) = \beta\big(F_0(A), F_0(A)\big)_\sim$$

is the well known Duhamel two point function, or canonical correlation. In the sequel, c_A will act as a *quantization parameter* or Planck's constant for the normal modes corresponding to the fluctuations of A.

The assumption on the spectral measure of the fluctuations of A then amounts to the assumption that there exists $\epsilon_A > 0$ such that

$$dc_A(\lambda) = c_A \delta(\lambda - \epsilon_A) d\lambda. \tag{10}$$

The "position" operator $Q_0(A)$ and "momentum" operator $P_0(A)$ of the normal mode are now defined by

$$Q_0(A) \equiv F_0(A), \qquad P_0(A) \equiv F_0(i\tilde{h}^{-1}A).$$

Obviously $P_0(A)$ is well defined because of the assumption (10).

The following proposition justifies the name *normal mode*:

Proposition 4. *The pair* $\big(Q_0(A), P_0(A)\big)$ *forms a quantum canonical pair,*

$$\big[Q_0(A), P_0(A)\big] = ic_A,$$

satisfying the equations of motion of a free quantum harmonic oscillator with frequency ϵ_A*:*

$$\tilde{\alpha}_t Q_0(A) = Q_0(A) \cos \epsilon_A t + \epsilon_A P_0(A) \sin \epsilon_A t,$$
$$\tilde{\alpha}_t P_0(A) = -\frac{1}{\epsilon_A} Q_0(A) \sin \epsilon_A t + P_0(A) \cos \epsilon_A t.$$

The $(\tilde{\alpha}_t, \beta)$*-KMS property of* $\tilde{\omega}$ *is expressed by*

$$\tilde{\omega}\left(Q_0(A)^2\right) = \epsilon_A^2 \tilde{\omega}\left(P_0(A)^2\right) = \frac{c_A \epsilon_A}{2} \coth \frac{\beta \epsilon_A}{2}.$$

Proof. By the KMS property of $\tilde{\omega}$,

$$\sigma\big(F_0(A), F_0(i\tilde{h}^{-1}A)\big) = \int (1 - e^{-\beta\lambda})\lambda^{-1} d\tilde{\mu}_A(\lambda) = c_A.$$

Lemma 3 and assumption (10) yield

$$\tilde{\omega}\left(Q_0(A)^2\right) = \epsilon_A^2 \tilde{\omega}\left(P_0(A)^2\right) = \frac{c_A \epsilon_A}{2} \coth \frac{\beta\epsilon_A}{2}.$$

A similar computation yields $\langle \epsilon_A JA - i\tilde{h}A, \epsilon_A JA - i\tilde{h}A\rangle_A = 0$, and by the equivalence relation (Eq. (5)), $F_0(i\tilde{h}A) = \epsilon_A F_0(JA)$, and by exponentiation:

$$\tilde{\alpha}_t F_0(A) = F_0(e^{\epsilon_A t J} A);$$

$J^2 = -1$ yields

$$\tilde{\alpha}_t F_0(A) = F_0(A) \cos \epsilon_A t + F_0(JA) \sin \epsilon_A t.$$

As above one shows that by the equivalence relation (5),

$$F_0(i\tilde{h}^{-1}A) = \epsilon_A^{-1} F_0(JA)$$

yielding the equations of motion as stated in the proposition. □

The creation and annihilation operators corresponding to this harmonic mode are simply the creation and annihilation operators defined in (9), although it is customary to rescale them with $\sqrt{\epsilon_A}$, i.e.

$$\frac{1}{\sqrt{\epsilon_A}} a_0^{\pm}(A) = \frac{Q_0(A) \mp i\epsilon_A P_0(A)}{\sqrt{2\epsilon_A}}.$$

Let us now consider how this situation can be extended to the more general case where the measure $d\tilde{\mu}_A(\lambda)$ has some spectral support Δ_A (see (6)). To avoid problems at energy $\lambda = 0$, we assume Δ_A to be bounded away from 0, i.e. there exists $\epsilon_A > 0$ such that

$$\Delta_A^+ \equiv \Delta_A \cap \mathbb{R}^+ \subseteq [\epsilon_A, +\infty).$$

Remark that Δ_A^+ is the support of the measure $dc_A(\lambda)$. In this case we can safely assume this measure to be absolutely continuous, i.e.

$$dc_A(\lambda) = c_A(\lambda) d\lambda.$$

Lemma 3 yields

$$\tilde{\omega}\big(F_0(A)^2\big) = \int_{\Delta_A^+} \frac{c_A(\lambda)\lambda}{2} \coth \frac{\beta\lambda}{2} d\lambda.$$

It is easily seen that instead of a single mode $\big(Q_0(A), P_0(A)\big)$ one can construct in this situation a continuous family of harmonic modes, i.e. two operator valued distributions

$$\left\{\big(Q_{0,A}(\lambda), P_{0,A}(\lambda)\big) \mid \lambda \in \Delta_A^+\right\},$$

such that

$$\begin{aligned}
\left[Q_{0,A}(\lambda), P_{0,A}(\lambda')\right] &= ic_A(\lambda)\delta(\lambda-\lambda'),\\
\tilde{\omega}\big(Q_{0,A}(\lambda)^2\big) = \lambda^2\tilde{\omega}\big(P_{0,A}(\lambda)^2\big) &= \frac{c_A(\lambda)\lambda}{2}\coth\frac{\beta\lambda}{2},\\
\tilde{\alpha}_t Q_{0,A}(\lambda) &= Q_{0,A}(\lambda)\cos\lambda t + \lambda P_{0,A}(\lambda)\sin\lambda t,\\
\tilde{\alpha}_t P_{0,A}(\lambda) &= -\frac{1}{\lambda}Q_{0,A}(\lambda)\sin\lambda t + P_{0,A}(\lambda)\cos\lambda t.
\end{aligned}$$

One identifies

$$\begin{aligned}
F_0(A) = Q_0(A) &= \int_{\Delta_A^+} Q_{0,A}(\lambda)d\lambda\\
F_0(i\tilde{h}^{-1}A) = P_0(A) &= \int_{\Delta_A^+} P_{0,A}(\lambda)d\lambda.
\end{aligned}$$

Remark that due to the spectral gap $P_0(A)$ is well defined and that by the spectral theory [18], $Q_{0,A}(\lambda)$ can be arbitrarily well approximated [16, Proposition 3.2.40] by a sequence of operators $F_0(A(f_i))$, where $\hat{f}_i \in \mathcal{D}$ is a sequence converging to a double δ-peak in $\pm\lambda$.

The content of this paper is to apply the construction of Proposition 4 to the situation of spontaneous breaking of a continuous symmetry, where we take for A the symmetry generator (i.e. the "charge" operator). The normal modes corresponding to the fluctuations of the symmetry generator as constructed above then yield a rigorous mathematical representation of the collective modes accompanying the spontaneous symmetry breaking (SSB), i.e. of the *Goldstone bosons*.

There are two distinct situations to consider, either the system with SSB has a gap in the energy spectrum, or it has not. The former situation is typically connected with long range interactions, the latter with short range interactions. Both situations introduce specific problems that make Proposition 4 not directly applicable as such.

Long range interacting systems in general do not possess a well-defined time evolution in the thermodynamic limit. Therefore one is restricted to studying specific models. In Sect. 3 we study a prototype model of a long range interacting system with a well-defined time evolution and a spectral gap, i.e. a mean field system. These systems have normal fluctuations, hence one can apply Proposition 4 directly.

The presence of SSB in short range interacting systems is characterized by either bad clustering properties (for temperature $T > 0$) or the absence of a spectral gap ($T = 0$). This is the content of the Goldstone Theorem (see Sect. 4 and references [19,20] for more details). Therefore these systems do not have normal fluctuations as defined in this section, i.e. there is *off diagonal long range order* in the system. For the systems we are interested in, this is a statement that applies to momentum $k = 0$ only, and one goes around this problem by working with the k-mode fluctuations, $k \neq 0$,

$$F_{n,k}(A) = \frac{1}{|\Lambda_n|^{1/2}}\sum_{x\in\Lambda_n}\big(\tau_x A - \omega(A)\big)\cos k.x.$$

These fluctuation operators will be shown to be normal and it will also be shown that in the ground state ($T = 0$) one can recover the situation of Proposition 4 in a properly scaled limit $k \to 0$. This is the content of Sect. 4.

3. Long Range Interactions

3.1. Introduction. In this section we study symmetry breaking systems whose Hamiltonian has a gap in the ground state. These systems typically have long range interactions, but since there is no general criterium whether a long range interacting system has a spectral gap or not, and since an infinite volume time evolution in general may not exist for these systems (see condition (1)), we restrict ourself to mean field systems which are long range interacting systems with a well defined time evolution in the thermodynamic limit and with a spectral gap. For the sake of clarity we consider an explicit example, namely the strong coupling BCS-model for superconductivity. Similar results as the ones presented here have already been obtained for different other mean field models [7,8], and for the jellium model [10], albeit by different methods. Moreover our main contribution in this section is *the construction of a canonical order parameter*.

The Hamiltonian for the strong coupling BCS-model is given by [21,22]

$$H_N = \epsilon \sum_{i=-N}^{N} \sigma_i^z - \frac{1}{2N+1} \sum_{i,j=-N}^{N} \sigma_i^+ \sigma_j^-, \quad \epsilon < \frac{1}{2},$$

where σ^z, $\sigma^\pm$ are the usual (2×2) Pauli matrices. H_N acts on the Hilbert space $\otimes_{i=-N}^{N} \mathbb{C}_i^2$.

The solutions of the KMS equation are given by the product states $\omega_\lambda = \omega_{\rho_\lambda}$ on the infinite tensor product algebra $\mathcal{A} = \otimes_{i=-\infty}^{\infty} (M_2)_i$ of the system; ρ_λ is a (2×2) density matrix, given by the solutions of the gap equation

$$\rho_\lambda = \frac{e^{-\beta h_\lambda}}{\mathrm{tr}\, e^{-\beta h_\lambda}}, \quad \lambda = \mathrm{tr}\, \rho_\lambda \sigma^- = \omega_\lambda(\sigma^-), \quad h_\lambda = \epsilon \sigma^z - \lambda \sigma^+ - \bar{\lambda} \sigma^-.$$

This is easily turned into the equation for λ:

$$\lambda \left(1 - \frac{\tanh \beta\mu}{2\mu}\right) = 0 \tag{11}$$

with $\mu = (\epsilon^2 + |\lambda|^2)^{1/2}$. Clearly, this equation has always the solution $\lambda = 0$, describing the so-called normal phase. We are interested in the solutions $\lambda \neq 0$ which exist in the case $\beta > \beta_c$ where β_c is determined by the equation $\tanh \beta_c \epsilon = 2\epsilon$. These solutions $\lambda \neq 0$ are understood to describe the superconducting phase. Remark that if $\lambda \neq 0$ is a solution of (11), then for all $\phi \in [0, 2\pi)$, $\lambda e^{i\phi}$ is a solution as well. There is an infinite degeneracy of the states for the superconducting phase. The degeneracy is due to the breaking of the gauge symmetry. As $\sigma^z = \sigma^+\sigma^- - \sigma^-\sigma^+$ it is clear that the Hamiltonian H_N is invariant under the continuous gauge transformations automorphism group $\mathcal{G} = \{\gamma_\phi | \phi \in [0, 2\pi)\}$ of $\mathcal{A}$,

$$\gamma_\phi : \sigma_i^+ \to \gamma_\phi(\sigma_i^+) = e^{-i\phi} \sigma_i^+.$$

However the solutions ω_λ are not invariant for this symmetry transformation, because:

$$\omega_\lambda(\gamma_\phi(\sigma_i^+)) = e^{-i\phi} \omega_\lambda(\sigma_i^+) \neq \omega_\lambda(\sigma_i^+). \tag{12}$$

The gauge symmetry of the system is spontaneously broken. Remark that h_λ is no longer invariant under the symmetry transformation, this is a typical feature of long range interacting systems. From (12) it follows also that $\omega_\lambda \circ \gamma_\phi = \omega_{\lambda e^{i\phi}}$, i.e. one solution ω_λ is transformed into another solution $\omega_{\lambda e^{i\phi}}$ by the gauge transformation γ_ϕ.

The gauge group $\mathcal{G}$ is not implemented by unitaries in any of the representations induced by the solutions ω_λ. Locally however, the gauge transformation γ_ϕ is implemented by unitaries: take any finite set Λ of indices, then

$$\gamma_\phi\Big(\prod_{i\in\Lambda}\sigma_i^-\Big) = (U_\phi^\Lambda)^*\Big(\prod_{i\in\Lambda}\sigma_i^-\Big)U_\phi^\Lambda,$$

where

$$U_\phi^\Lambda = e^{\frac{i}{2}\phi Q_\Lambda}, \quad Q_\Lambda = \sum_j \sigma_j^z.$$

The operator Q_Λ is called the local charge or symmetry generator and σ^z the charge density or symmetry generator density.

3.2. Canonical coordinates of the Goldstone mode. Next we introduce the algebra of fluctuations and show how the Goldstone mode operators are to be defined in a canonical way. The relation between symmetry breaking and quantum fluctuations in the strong coupling BCS model has been studied before in [6]. This analysis is here extended.

Per lattice site $j \in \mathbb{Z}$ one has the local algeba of observables, the real (2×2) matrices, M_2, generated by the Pauli matrices. As state we consider a particular equilibrium state ω_λ with $\beta > \beta_c$ which reduces per lattice point to the trace state $\omega_\lambda(A) = \mathrm{tr}\,\rho_\lambda A$, $A \in M_2$. Because of the product character of the algebra, the state and the time evolution, it is sufficient to consider fluctuations of one-point observables. Locally the fluctuation of A in the state ω_λ is:

$$F_N(A) = \frac{1}{(2N+1)^{1/2}}\sum_{i=-N}^{N}\big(A_i - \rho_\lambda(A)\big), \quad A \in M_2.$$

The commutator of two fluctuations is a mean, indeed:

$$\big[F_N(A), F_N(B)\big] = \frac{1}{2N+1}\sum_{i=-N}^{N}\big([A, B]\big)_i.$$

For $A, B \in M_2$ define

$$s_\lambda(A, B) = \mathrm{Re}\ \rho_\lambda\Big(\big(A - \rho_\lambda(A)\big)\big(B - \rho_\lambda(B)\big)\Big),$$
$$\sigma_\lambda(A, B) = \mathrm{Im}\ \rho_\lambda\Big(\big[A - \rho_\lambda(A)\big]\big[B - \rho_\lambda(B)\big]\Big) = -i\rho_\lambda\big([A, B]\big).$$

Clearly $(M_{2,sa}, \sigma_\lambda)$ is a symplectic space and s_λ is a symmetric positive bilinear form on $M_{2,sa}$.

Because ρ_λ is time invariant, $\rho_\lambda \circ \alpha_t = \rho_\lambda$ and because the evolution α_t is local, $\alpha_t : M_{2,sa} \to M_{2,sa}$, one has that α_t is a symplectic operator on $(M_{2,sa}, \sigma_\lambda)$: for all $t \in \mathbb{R}$,

$$\sigma_\lambda(\alpha_t A, \alpha_t B) = \sigma_\lambda(A, B).$$

The structure $(M_{2,sa}, \sigma_\lambda, s_\lambda, \alpha_t)$ defines in a canonical way the CCR-dynamical system $\left(\overline{W(M_{2,sa}, \sigma_\lambda)}, \tilde{\omega}_\lambda, \tilde{\alpha}_t\right)$; $\tilde{\omega}_\lambda$ is a quasi-free state on the CCR-algebra $\overline{W(M_{2,sa}, \sigma_\lambda)}$:

$$\tilde{\omega}_\lambda\big(W(A)\big) = e^{-\frac{1}{2}s_\lambda(A,A)} \quad \text{and} \quad \tilde{\alpha}_t\big(W(A)\big) = W\big(\alpha_t(A)\big)$$

for all $A \in M_{2,sa}$.

Let $(\tilde{\mathcal{H}}_\lambda, \tilde{\pi}_\lambda, \tilde{\Omega}_\lambda)$ be the GNS triplet of $\tilde{\omega}_\lambda$. As the state $\tilde{\omega}_\lambda$ is regular, there exists a real linear map, called the bose field $F_\lambda : M_{2,sa} \to \mathcal{L}(\tilde{\mathcal{H}}_\lambda)$ such that $\tilde{\pi}_\lambda\big(W(A)\big) = e^{iF_\lambda(A)}$ and the commutation relations $\big[F_\lambda(A), F_\lambda(B)\big] = i\sigma_\lambda(A, B)$. As in Sect. 2.2, a central limit theorem allows the identification $\lim_{N\to\infty} F_N(A) = F_\lambda(A)$. The state $\tilde{\omega}_\lambda$ is completely characterized by the two-point function on the algebra of fluctuations

$$\tilde{\omega}_\lambda\big(F_\lambda(A)F_\lambda(B)\big) = \lim_{N\to\infty} \omega_\lambda\big(F_N(A)F_N(B)\big) = s_\lambda(A, B) + \frac{i}{2}\sigma_\lambda(A, B).$$

Now we proceed to the construction of the complex structure J (see Sect. 2.4). By diagonalisation of the matrix h_λ it is easily seen that h_λ has eigenvalues $\pm\mu$, where $\mu = (\epsilon^2 + |\lambda|^2)^{1/2}$. The spectral resolution of h_λ is hence given by

$$h_\lambda = -\mu P_- + \mu P_+.$$

In order to construct J we need to know the spectral resolution of $[h_\lambda, \cdot]$ considered as operator on M_2. The spectrum of $[h_\lambda, \cdot]$ is given by $\{-2\mu, 0, 2\mu\}$, the corresponding spectral projections are respectively:

$$E_- = E(-2\mu) = P_- \cdot P_+, \quad E_0 = P_- \cdot P_- + P_+ \cdot P_+, \quad E_+ = E(2\mu) = P_+ \cdot P_-,$$

and $[h_\lambda, A] = -2\mu E_-(A) + 2\mu E_+(A)$.

On $M^1_{2,sa} \equiv (E_+ + E_-)M_{2,sa}$ define J as in Sect. 2 (Eq. (8)) by

$$J(E_+ + E_-)(A) = i(E_+ - E_-)(A).$$

This operator J is a complex structure on the symplectic space $(M^1_{2,sa}, \sigma_\lambda)$, satisfying the properties of Proposition 2: $J^2 = -1$, $\sigma_\lambda(A, JB) = -\sigma_\lambda(JA, B)$, $A, B \in M^1_{2,sa}$ and $\sigma_\lambda(A, JA) > 0$, if $0 \neq A \in M^1_{2,sa}$. Remark that on $M^1_{2,sa}$, $[h_\lambda, \cdot] = -2i\mu J(\cdot)$ (cf. Proposition 4).

For $\lambda \neq 0$, we have $[h_\lambda, \sigma^z] \neq 0$. However $[h_\lambda, E_0(\sigma^z)] = 0$, and the state ω_λ and the corresponding time evolution α_t are still invariant under the symmetry generated by $E_0(\sigma^z)$:

$$\lim_{N\to\infty} \omega_\lambda\left(\Big[\sum_{i=-N}^{N} E_0(\sigma^z)_i, A\Big]\right) = 0$$

for all local A. Symmetry breaking is only concerned with the operator

$$\hat{\sigma}^z \equiv \sigma^z - E_0(\sigma^z) = (E_+ + E_-)(\sigma^z);$$

$\hat{\sigma}^z \in M^1_{2,sa}$ and we are interested in the fluctuations of the operator $\hat{\sigma}^z$ together with its adjoint $J\hat{\sigma}^z$. By calculating $[h_\lambda, \sigma^z] = 2\mu(E_+ - E_-)(\sigma^z)$, we find

$$J\hat{\sigma}^z = \frac{i}{\mu}(\lambda\sigma^+ - \bar{\lambda}\sigma^-).$$

Similarly $[h_\lambda, J\hat{\sigma}^z] = 2i\mu(E_+ + E_-)(\sigma^z)$ yields

$$\hat{\sigma}^z = \frac{|\lambda|^2}{\mu^2}\sigma^z + \frac{\epsilon}{\mu^2}(\lambda\sigma^+ + \bar{\lambda}\sigma^-).$$

Note that $J\hat{\sigma}^z$ is the usual order parameter operator for the BCS model, but now constructed by means of σ^z and the spectrum of the Hamiltonian. Therefore it is called the *canonical order parameter operator*. We have also $\omega_\lambda\big(J\hat{\sigma}^z\big) = 0$ and $0 = \omega_\lambda\big([h_\lambda, J\hat{\sigma}^z]\big) = 2i\mu\omega_\lambda(\hat{\sigma}^z)$.

The variances of the fluctuation operators are easily calculated since

$$(E_0\sigma^z)^2 = \frac{\epsilon^2}{\mu^2}, \quad (\hat{\sigma}^z)^2 = (J\hat{\sigma}^z)^2 = \frac{|\lambda|^2}{\mu^2}.$$

Note $1 = (\sigma^z)^2 = E_0(\sigma^z)^2 + (\hat{\sigma}^z)^2$. Also

$$\rho_\lambda(\sigma^z) = \rho_\lambda(E_0\sigma^z) = -\frac{\epsilon}{\mu}\tanh\beta\mu = -2\epsilon.$$

Hence

$$\begin{aligned}\tilde{\omega}_\lambda\big(F_\lambda(E_0\sigma^z)^2\big) &= s_\lambda(E_0\sigma^z, E_0\sigma^z)\\ &= \rho_\lambda\big((E_0\sigma^z)^2\big) - \rho_\lambda(E_0\sigma^z)^2 = \frac{\epsilon^2}{\mu^2} - 4\epsilon^2,\\ \tilde{\omega}_\lambda\big(F_\lambda(\hat{\sigma}^z)^2\big) &= s_\lambda(\hat{\sigma}^z, \hat{\sigma}^z) = \rho_\lambda\big((\hat{\sigma}^z)^2\big) = \frac{|\lambda|^2}{\mu^2},\\ \tilde{\omega}_\lambda\big(F_\lambda(J\hat{\sigma}^z)^2\big) &= s_\lambda(J\hat{\sigma}^z, J\hat{\sigma}^z) = \rho_\lambda\big((J\hat{\sigma}^z)^2\big) = \frac{|\lambda|^2}{\mu^2}.\end{aligned}$$

The only non-trivial commutator is

$$\begin{aligned}\big[F_\lambda(\hat{\sigma}^z), F_\lambda(J\hat{\sigma}^z)\big] &= i\sigma_\lambda(\hat{\sigma}^z, J\hat{\sigma}^z) = \omega_\lambda\big([\hat{\sigma}^z, J\hat{\sigma}^z]\big)\\ &= \omega_\lambda\big([\sigma^z, J\hat{\sigma}^z]\big) = i\,\frac{4|\lambda|^2}{\mu},\end{aligned}$$

expressing the bosonic character of the fluctuations. Remark on the other hand that the microscopic observables $\hat{\sigma}^z$ and $J\hat{\sigma}^z$ do not satisfy canonical commutation relations, only their fluctuations do.

The flucuation operator $F_\lambda(E_0\sigma^z)$ is invariant under the dynamics $\tilde{\alpha}_t$, but the operators $F_\lambda(\hat{\sigma}^z)$ and $F_\lambda(J\hat{\sigma}^z)$ satisfy the equations of motion

$$\frac{d}{idt}\tilde{\alpha}_t\big(F_\lambda(\hat{\sigma}^z)\big) = F_\lambda\big([h_\lambda, \alpha_t(\hat{\sigma}^z)]\big) = -2i\mu F_\lambda\big(\alpha_t(J\hat{\sigma}^z)\big) = -2i\mu\tilde{\alpha}_t F_\lambda(J\hat{\sigma}^z), \tag{13}$$

$$\frac{d}{idt}\tilde{\alpha}_t\big(F_\lambda(J\hat{\sigma}^z)\big) = F_\lambda\big([h_\lambda, \alpha_t(J\hat{\sigma}^z)]\big) = 2i\mu F_\lambda\big(\alpha_t(\hat{\sigma}^z)\big) = 2i\mu\tilde{\alpha}_t F_\lambda\big(\hat{\sigma}^z\big). \tag{14}$$

In integrated form one gets:

$$\begin{aligned}\tilde{\alpha}_t F_\lambda(\hat{\sigma}^z) &= F_\lambda(\hat{\sigma}^z)\cos 2\mu t + F_\lambda(J\hat{\sigma}^z)\sin 2\mu t,\\ \tilde{\alpha}_t F_\lambda(J\hat{\sigma}^z) &= -F_\lambda(\hat{\sigma}^z)\sin 2\mu t + F_\lambda(J\hat{\sigma}^z)\cos 2\mu t.\end{aligned}$$

Hence by an explicit calculation we have arrived at the results of Proposition 4, for $A = \hat{\sigma}^z$, the generator of the broken symmetry. Therefore, denoting $Q_\lambda \equiv F_\lambda(\hat{\sigma}^z)$ and $P_\lambda \equiv \frac{1}{2\mu} F_\lambda(J\hat{\sigma}^z)$, we defined the pair (Q_λ, P_λ) as *the canonical pair of the Goldstone bosons*. Writing down the previous results in terms of Q_λ and P_λ (as in Proposition 4) one sees that this pair shares indeed all physical properties for Goldstone bosons.

Remark that the frequency of oscillation is 2μ. This is the phenomenon of the doubling of the frequency for the inherent plasmon frequency.

The formula

$$\tilde{\omega}_\lambda\left(Q_\lambda^2\right) = (2\mu)^2 \tilde{\omega}_\lambda\left(P_\lambda^2\right) = \frac{|\lambda|^2}{\mu^2} = \frac{c_\lambda(2\mu)}{2} \coth \frac{\beta(2\mu)}{2},$$

is a quantum mechanical expression of a virial theorem. Remark that in the normal phase $(\lambda \to 0)$, $Q_{\lambda=0} = P_{\lambda=0} = 0$, i.e. the Goldstone boson disappears.

The creation and annihilation operators of the Goldstone bosons are as usual

$$a_\lambda^\pm = \frac{Q_\lambda \mp i2\mu P_\lambda}{\sqrt{4\mu}}.$$

The state $\tilde{\omega}_\lambda$ is gauge-invariant and quasi-free with respect to the gauge transformations of these creation and annihilation operators, i.e. $\tilde{\omega}_\lambda\left(a_\lambda^+ a_\lambda^+\right) = 0 = \tilde{\omega}_\lambda\left(a_\lambda^+\right)$, and the two-point function

$$\tilde{\omega}_\lambda\left(a_\lambda^+ a_\lambda^-\right) = \frac{1}{e^{2\beta\mu} - 1}.$$

4. Short Range Interactions

4.1. Goldstone theorem and canonical order parameter. Let ω be an extremal translation invariant (α_t, β)-KMS state, α_t a dynamics generated by a translation invariant Hamiltonian H and let γ_s be a strongly continuous one-parameter symmetry group which is locally generated by a generator

$$Q_n = \sum_{x \in \Lambda_n} q_x,$$

where $\Lambda_n = [-n, n]^\nu \cap \mathbb{Z}^\nu$ and q_x is the symmetry generator density, i.e. for $A \in \mathcal{A}_{\Lambda_n}$,

$$\gamma_s(A) = e^{isQ_n} A\, e^{-isQ_n}.$$

Denote $q = q_{x=0}$, and for convenience denote again $q - \omega(q)$ by q.

For systems with short range interactions, assuming spontaneous symmetry breaking amounts to:

Assumption 1. *Assume that there exists an (α_t, β)-KMS or ground state ω such that ω is not invariant under the symmetry transformation γ, while the dynamics α_t remains invariant under γ, i.e.*

$$\exists A \in \mathcal{A}_L \;\; \textit{such that} \;\; \omega\big(\gamma_s(A)\big) \neq \omega(A), \tag{15}$$

$$\alpha_t \circ \gamma_s = \gamma_s \circ \alpha_t. \tag{16}$$

The invariance of the dynamics (16) is crucial in this context (see [23] and Proposition 6 and Eq. (24) below). For a more complete discussion of the phenomenon of spontaneous symmetry breaking, see [20].

An operator A satisfying (15) is called an order parameter operator. Equation (15) is equivalent to

$$\frac{d}{ds}\omega\big(\gamma_s(A)\big)\Big|_{s=0} = \lim_{n\to\infty}\omega\big([Q_n, A]\big) \neq 0.$$

The local Hamiltonians are determined by an interaction Φ

$$H_n = \sum_{X\subseteq\Lambda_n} \Phi(X)$$

and the infinite volume Hamiltonian H is defined such that for $A \in \mathcal{A}_{\Lambda_0}$,

$$HA\Omega = \sum_{X\cap\Lambda_0\neq\varnothing} [\Phi(X), A]\Omega,$$

where Ω is the cyclic vector of the state ω.

The relation between spontaneous symmetry breaking and the absence of a gap in the energy spectrum in the ground state was originally put forward by Goldstone [1]. For short range interactions in many-body systems, it is proved [2,3] that spontaneous symmetry breaking implies the absence of an energy gap in the excitation spectrum. We refer here to [19] where the Goldstone theorem is proved rigorously for quantum lattice systems.

Theorem 5 (Goldstone Theorem [19]). *If Φ is translation invariant and satisfies*

$$\sum_{X\ni 0} |X|\|\Phi(X)\| < \infty, \tag{17}$$

then

(i) *At $T = 0$: If the system has an energy gap then there is no spontaneous symmetry breakdown.*
(ii) *At $T > 0$: If the system has L^1 clustering then there is no spontaneous symmetry breakdown.*

The L^1 clustering means here that for each observable A, one has:

$$\sum_{x\in\mathbb{Z}^\nu} \left|\omega(A\tau_x A) - \omega(A)^2\right| < \infty.$$

The first step is to construct something like a *canonical order parameter operator*. See Sect. 3 for an example of this construction. Denote

$$L(A) = [H, A].$$

The Duhamel two-point function becomes now:

$$(A, B)_\sim \equiv \frac{1}{\beta}\int_0^\beta \omega(A^*\alpha_{iu}B)du = \omega\left(A^*\frac{1-e^{-\beta L}}{\beta L}B\right).$$

The KMS-condition, $\omega(AB) = \omega\left(B\alpha_{i\beta}A\right)$, yields

$$\omega\big([A, B]\big) = \omega\Big(A(1 - e^{-\beta L})B\Big),$$

for A, B in a dense domain of $\mathcal{A}$, and hence if $B \in \mathrm{Dom}(L^{-1})$ then

$$\beta(A, B)_\sim = \omega\Big(\big[A, L^{-1}B\big]\Big),$$

and the Bogoliubov inequality [24] for KMS-states is given by:

$$\big|\omega\big([A^*, B]\big)\big|^2 \leq \beta\omega\big([A^*, L(A)]\big)(B, B)_\sim.$$

Finally denote the local 0-mode fluctuation of an observable A in the state ω by

$$F_{n,0}(A) = \frac{1}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n} \tau_x A - \omega(A).$$

Assumption 2. *Assume that there are no long range correlations in the fluctuations of the symmetry generator density, i.e. assume*

$$\lim_{n\to\infty} \omega\big(F_{n,0}(q)^2\big) = \sum_{z\in\mathbb{Z}^\nu} \Big|\omega(q\tau_z q) - \omega(q)^2\Big| < \infty.$$

Then also the uniform susceptibility c_0^β defined by

$$c_0^\beta \equiv \lim_{n\to\infty} \frac{\beta}{2}\big(F_{n,0}(q), F_{n,0}(q)\big)_\sim \tag{18}$$

is finite, i.e. $c_0^\beta < \infty$.

Proposition 6. *Under Assumption 1 and 2 we have*

$$c_0^\beta = \lim_{n\to\infty} \frac{\beta}{2}\big(F_{n,0}(q), \alpha_t F_{n,0}(q)\big)_\sim > 0 \tag{19}$$

and c_0^β is independent of t, and given by

$$c_0^\beta = \lim_{n\to\infty} \frac{1}{2}\omega\Big(\big[Q_n, L^{-1}(q)\big]\Big).$$

Proof. Let

$$\begin{aligned} c_0^\beta(t) &= \lim_{n\to\infty} \frac{\beta}{2}\big(F_{n,0}(q), \alpha_t F_{n,0}(q)\big)_\sim \\ &= \lim_{n\to\infty} \frac{1}{2}\omega\Big(F_{n,0}(q)\frac{1 - e^{-\beta L}}{L} e^{itL} F_{n,0}(q)\Big). \end{aligned}$$

First we show $c_0^\beta(t = 0) > 0$. Let A be an arbitrary order parameter operator. SSB, translation invariance and the Bogoliubov inequality yield

$$\begin{aligned} 0 &< \lim_{n\to\infty} \Big|\omega\big(\big[F_{n,0}(q), F_{n,0}(A)\big]\big)\Big|^2 \\ &\leq \lim_{n\to\infty} \beta\omega\Big(\big[F_{n,0}(A), L\big(F_{n,0}(A)\big)\big]\Big)\big(F_{n,0}(q), F_{n,0}(q)\big)_\sim. \end{aligned}$$

In [19] it is shown that (17) also implies

$$\lim_{n\to\infty} \omega\Big(\big[F_{n,0}(A), L\big(F_{n,0}(A)\big)\big]\Big) = \sum_{z\in\mathbb{Z}^\nu} \omega\Big(\big[\tau_z A, L(A)\big]\Big) < \infty$$

for each local observable A. Hence

$$0 < \sum_{z\in\mathbb{Z}^\nu} \omega\Big(\big[\tau_z A, L(A)\big]\Big) \lim_{n\to\infty} \beta\big(F_{n,0}(q), F_{n,0}(q)\big)_\sim$$

yielding $c_0^\beta(t=0) > 0$.

The proof of the time invariance of c_0^β is based on [23] and goes as follows:

$$\begin{aligned}
\frac{d}{idt} c_0^\beta(t) &= \lim_{n\to\infty} \frac{\beta}{2}\Big(F_{n,0}(q), \alpha_t L\big(F_{n,0}(q)\big)\Big)_\sim \\
&= \lim_{n\to\infty} \frac{1}{2}\omega\Big(F_{n,0}(q)(1-e^{-\beta L})e^{itL} F_{n,0}(q)\Big) \\
&= \lim_{n\to\infty} \frac{1}{2}\omega\Big(\big[F_{n,0}(q), e^{itL} F_{n,0}(q)\big]\Big).
\end{aligned}$$

Translation invariance and (16) yield:

$$\begin{aligned}
\frac{d}{idt} c_0^\beta(t) &= \lim_{n\to\infty} \frac{1}{2}\omega\Big(\big[Q_n, e^{itL} q\big]\Big) \\
&= \frac{1}{2} \left.\frac{d}{ids}\right|_{s=0} \omega\big(\gamma_s(\alpha_t q)\big) = \frac{1}{2} \left.\frac{d}{ids}\right|_{s=0} \omega\big(\alpha_t(\gamma_s q)\big) \\
&= \frac{1}{2} \left.\frac{d}{ids}\right|_{s=0} \omega(q) = 0. \qquad \square
\end{aligned}$$

From the proposition it follows that if $L^{-1}q$ exists, it is an order parameter operator. We call it the *canonical order parameter operator*, it is an order parameter constructed directly from the two given quantities, the Hamiltonian and the symmetry generator. However it can not be expected in general that $q \in \mathrm{Dom}(L^{-1})$, especially not for systems without an energy gap, because of problems at zero energy. Expressions like $(1-e^{-\beta L})L^{-1}q$ on the contrary are well defined. The bulk of our efforts below consists of mastering the difficulties with the canonical order parameter by considering the k-mode fluctuations and by afterwards taking the limit $k \to 0$. This method has already been used in [15], where the Goldstone coordinates are constructed for models of interacting Bose gases.

4.2. Fluctuations. By the Goldstone theorem, spontaneous symmetry breaking implies that the system does not have exponential or L^1 clustering. In particular the variances of local fluctuations $F_{n,0}(A)$ may not be convergent in the thermodynamic limit for certain A (in particular for A an order parameter operator) because of long range order correlations. The central limit as described in Sect. 2.2 no longer holds. However one can study the k-mode fluctuations, i.e. one considers for $k = (k_1, k_2, \ldots, k_\nu) \in \mathbb{R}^\nu$, with $k_j \neq 0$ for $j = 1, 2, \ldots, \nu$:

$$F_{n,k}(A) = \frac{1}{|\Lambda_n|^{1/2}} \sum_{x\in\Lambda_n} \big(\tau_x(A) - \omega(A)\big) \cos k.x.$$

It is believed that the central limit theorem holds for the k-mode fluctuations in every extremal translation invariant state, even at criticality. This is essentially because one stays away from the singularity at $k = 0$. A completely rigorous proof of this statement is found in [25], for the absolute convergent case under a very mild cluster condition. Below we prove the convergence of the Fourier series for translation invariant states with singularities occuring only at zero momentum (see further on). See also [26] for a similar line of reasoning.

For $A \in \mathcal{A}_L$, denote the Fourier transforms of the l-point correlation functions $\omega(\tau_{x_1} A \tau_{x_2} A \cdots \tau_{x_l} A)$ by $\mu(k_1, k_2, \cdots, k_l)$ (i.e. k_j are different vectors in $\mathbb{R}^\nu$ here, not the components of a particular k). In general μ is a measure. By translation invariance it can be written as a function of $k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_l$. As in [26], assume that the only singularities in μ are of the type $\delta(k_1 + \cdots + k_i)$ (i.e. singularities occurring only at zero momentum).

We show now that the truncated correlation functions $\omega_T\left(F_{n,k}(A)^l\right)$ vanish for $l \geq 3$ and remain finite for $l = 2$. Let $\omega(A) = 0$, then

$$\begin{aligned}
&\omega_T\left(F_{n,k}(A)^l\right) \\
&= \frac{1}{|\Lambda_n|^{l/2}} \sum_{x_1, x_2, \ldots, x_l} \omega_T(\tau_{x_1} A \tau_{x_2} A \cdots \tau_{x_l} A) \cos k.x_1 \cos k.x_2 \cdots \cos k.x_l \\
&= \frac{1}{|\Lambda_n|^{l/2}} \sum_{x_1, y_1 \ldots, y_{l-1}} \omega_T(A \tau_{y_1} A \cdots \tau_{y_{l-1}} A) \cos k.x_1 \cos k.(y_1 + x_1) \\
&\qquad \cdots \cos k.(y_{l-1} + x_1).
\end{aligned}$$

The expansion of the cosines into exponentials yields two types of terms, namely terms which do not depend on x_1 and terms which do depend on x_1. The first kind of terms do not appear for l odd and for l even they are exactly the ones which are cancelled out by the truncation. The second kind of terms tend to zero because of the scaling factors. Let us illustrate this by means of an example. First let $l = 2$:

$$\omega_T\left(F_{n,k}(A)^2\right) = \omega\left(F_{n,k}(A)^2\right) = \frac{1}{|\Lambda_n|} \sum_{x,y} \omega(A \tau_{y-x} A) \cos k.x \cos k.y.$$

Since

$$\mu(k) = \sum_z \omega(A \tau_z A) e^{-ik.z}$$

can at most have a singularity at $k = 0$, $\mu(k) < \infty$ for $k \neq 0$. Also

$$\frac{1}{|\Lambda_n|} \sum_z e^{ik.z} \to \delta_{k,0}.$$

Hence the only terms contributing in the two-point correlation function are the terms containing the factor $e^{\pm ik.(y-x)}$, i.e. the terms of the first kind. In the limit we find

$$\lim_n \omega_T\left(F_{n,k}(A)^2\right) = \frac{1}{4}[\mu(k) + \mu(-k)] < \infty.$$

Now let $l = 4$ and consider a typical term:

$$\frac{1}{|\Lambda_n|^2} \sum_{x_1, y_1, y_2, y_3} \omega\left(A\tau_{y_1} A\tau_{y_2} A\tau_{y_3} A\right) e^{-ik.(y_1 - y_2 + y_3)}.$$

Ignoring boundary effects in the sums, this becomes

$$\begin{aligned} &\frac{1}{|\Lambda_n|} \sum_{y_1, y_2, y_3} \omega\left(A\tau_{y_1} A\tau_{y_2} A\tau_{y_3} A\right) e^{-ik.(y_1 - y_2 + y_3)} \\ &\quad = \frac{1}{|\Lambda_n|} \sum_{y_1, y_2, y_3} \omega\left(A\tau_{y_1} A\tau_{y_2} [A\tau_{y_3 - y_2} A]\right) e^{-ik.(y_1 - y_2 + y_3)} \\ &\quad = \sum_{x,z} \omega\Big(A\tau_x A \frac{1}{|\Lambda_n|} \sum_y \tau_y [A\tau_z A]\Big) e^{-ik.(x+z)}. \end{aligned}$$

In the limit we get

$$\sum_x \omega\left(A\tau_x A\right) e^{-ik.x} \sum_z \omega\left(A\tau_z A\right) e^{-ik.z},$$

cancelling out against two-point correlations in the 4-point truncated correlation function.

Finally, take $l = 3$, then all terms are of the second kind and vanish, e.g.

$$\begin{aligned} &\frac{1}{|\Lambda_n|^{3/2}} \sum_{x_1, y_1, y_2} \omega(A\tau_{y_1} A\tau_{y_2} A) e^{ik.(x_1 + y_1 - y_2)} \\ &\qquad\qquad = \frac{1}{|\Lambda_n|^{3/2}} \sum_{y_1, y_2} \omega(A\tau_{y_1} [A\tau_{y_2 - y_1} A]) e^{ik.(y_1 - y_2)} \sum_{x_1} e^{ik.x_1}. \end{aligned}$$

The sum over x_1 is bounded by $\prod_{j=1}^{\nu} |\sin \frac{k_j}{2}|^{-1}$, yielding

$$\sum_y \omega\Big(A \frac{1}{|\Lambda_n|} \sum_x \tau_x [A\tau_y A]\Big) e^{-ik.y}$$

which converges to

$$\omega(A) \sum_y \omega\left(A\tau_y A\right) e^{-ik.y} = 0.$$

Using the formula

$$\omega\left(e^{i\lambda Q}\right) = \exp\Big\{\sum_{l=1}^{\infty} \frac{(i\lambda)^l}{l!} \omega_T \big(\underbrace{Q, \ldots, Q}_{l \text{ times}}\big)\Big\},$$

one arrives at the central limit theorem

$$\lim_n \omega\left(e^{iF_{n,k}(A)}\right) = e^{-\frac{1}{2}s_k(A,A)},$$

with $s_k(A, A) = \lim_{n\to\infty} \omega\left(F_{n,k}(A)^2\right)$.

In [25] one can find a rigorous proof of the central limit theorem for the k-mode fluctuations, $k = (k_j \neq 0)_{j=1}^{\nu}$, for states satisfying a certain clustering condition, expressed as a condition on the function α_ω (see Eq. (3)). Although this condition is much weaker than for the $k = 0$ fluctuations, it is not clear whether it is always satisfied for any extremal translation invariant state. The arguments above however suggest that this clustering condition on the state is merely technical and that a general rigorous proof of the central limit theorem along the lines of [25] is possible for $k = (k_j \neq 0)_{j=1}^{\nu}$ under even weaker conditions. We continue on the basis of the arguments above.

Theorem 7 (Central limit theorem). *If the state ω has only singularities at zero momentum, for all $A \in \mathcal{A}_{L,sa}$ and $k = (k_j \neq 0)_{j=1}^{\nu}$, then*

(i) $\lim_{n\to\infty} \omega\left(F_{n,k}(A)^2\right) < \infty$,
(ii) $\lim_{n\to\infty} \omega\left(e^{iF_{n,k}(A)}\right) = e^{-\frac{1}{2}s_k(A,A)}$
with $s_k(A, B) = \lim_{n\to\infty} \mathrm{Re}\, \omega\left(F_{n,k}(A)^* F_{n,k}(B)\right)$.

Because of (i), the limit

$$\lim_{n\to\infty} \omega\Big(F_{n,k}(A)^* F_{n,k}(B)\Big) \equiv \langle A, B\rangle_k$$

defines a positive sesquilinear form which satisfies the Cauchy–Schwarz inequality

$$|\langle A, B\rangle_k|^2 \leq \langle A, A\rangle_k \langle B, B\rangle_k.$$

More explicitly

$$\langle A, B\rangle_k = \frac{1}{2} \sum_{z\in\mathbb{Z}^\nu} \Big(\omega(A^*\tau_z B) - \omega(A^*)\omega(B)\Big) \cos k.z.$$

Let

$$\sigma_k(A, B) = 2\ \mathrm{Im}\ \langle A, B\rangle_k,$$

then

$$\text{strong} - \lim_{n\to\infty} \pi\Big([F_{n,k}(A), F_{n,k}(B)]\Big) = i\sigma_k(A, B).$$

The identification of the central limit with bose fields is as in Sect. 2.2, and worked out in full detail for $k \neq 0$ in [25]. The bilinear form s_k determines a quasi free state $\tilde{\omega}_k$ on the CCR-algebra $\mathcal{W}(\mathcal{A}_{L,sa}, \sigma_k)$:

$$\tilde{\omega}_k\left(W_k(A)\right) = e^{-\frac{1}{2}s_k(A,A)}.$$

The $W_k(A)$, $A \in \mathcal{A}_{L,sa}$ are the Weyl operators generating $\mathcal{W}(\mathcal{A}_{L,sa}, \sigma_k)$. Via the central limit theorem, one shows for $A_1, A_2, \ldots, A_l \in \mathcal{A}_{L,sa}$,

$$\lim_{n\to\infty} \omega\Big(e^{iF_{n,k}(A_1)} e^{iF_{n,k}(A_2)} \ldots e^{iF_{n,k}(A_l)}\Big) = \tilde{\omega}_k\Big(W_k(A_1) W_k(A_2) \ldots W_k(A_l)\Big).$$

The state $\tilde{\omega}_k$ is regular and hence for every $A \in \mathcal{A}_{L,sa}$ there exists a self-adjoint bosonic field $F_k(A)$ in the GNS representation $(\tilde{\mathcal{H}}_k, \tilde{\pi}_k, \tilde{\Omega}_k)$ of $\tilde{\omega}_k$ such that

$$\tilde{\pi}_k(W_k(A)) = e^{iF_k(A)}.$$

This implies that in the sense of the central limit, the local fluctuations converge to the bosonic fields associated with the system $\left(\mathcal{W}(\mathcal{A}_{L,sa}, \sigma_k), \tilde{\omega}_k\right)$,

$$\lim_{n\to\infty} F_{n,k}(A) = F_k(A).$$

As in Sect. 2.2, fluctuation operators are only defined up to equivalence i.e. $A \equiv_k B$ if $\langle A - B, A - B\rangle_k = 0$ and

$$A \equiv_k B \Leftrightarrow \tilde{\pi}_k\big(W_k(A)\big) = \tilde{\pi}_k\big(W_k(B)\big). \tag{20}$$

The form $\langle\cdot,\cdot\rangle_k$ thus becomes a scalar product on $[\mathcal{A}_L]$, the equivalence classes of $\mathcal{A}_L$ for the relation $\equiv_k$. Denote by $\mathcal{K}_k$ the Hilbert space obtained as completion of $[\mathcal{A}_L]$ and by $\mathcal{K}_k^{\mathrm{Re}}$ the real subspace of $\mathcal{K}_k$ generated by $[\mathcal{A}_{L,sa}]$.

4.3. Goldstone modes for finite wavelengths. The finiteness of $\lim_{n\to\infty}\omega\big(F_{n,k}(q)^2\big)$ for all k ($k = 0$ included by Assumption 2) implies the finiteness of

$$\lim_{n\to\infty}\int |\hat{f}(\lambda)|\omega\big(F_{n,k}(q)dE_\lambda F_{n,k}(q)\big)$$

for $\hat{f} \in \mathcal{D}$, and hence the existence of a measure

$$d\tilde{\mu}_k(\lambda) = \lim_{n\to\infty}\omega\big(F_{n,k}(q)dE_\lambda F_{n,k}(q)\big);$$

dE_λ is the spectral measure of the Hamiltonian H, i.e. $H = \int \lambda dE_\lambda$.

As in Sect. 2.5, define the measure $dc_k^\beta(\lambda)$ with support on $\mathbb{R}^+$ only by

$$dc_k^\beta(\lambda) = 2\frac{1 - e^{-\beta\lambda}}{\lambda}d\tilde{\mu}_k(\lambda),$$

such that for $\hat{f} \in \mathcal{D}$ (cf. Lemma 3)

$$\lim_{n\to\infty}\int \hat{f}(\lambda)\omega\big(F_{n,k}(q)dE_\lambda F_{n,k}(q)\big) = \int_0^\infty \big(\hat{f}(\lambda) + \hat{f}(-\lambda)e^{-\beta\lambda}\big)\frac{\lambda}{2(1 - e^{-\beta\lambda})}dc_k^\beta(\lambda). \tag{21}$$

Proposition 8. *For $\hat{f} \in \mathcal{D}$,*

$$\lim_{k\to 0}\int_0^\infty \hat{f}(\lambda)dc_k^\beta(\lambda) = \int_0^\infty \hat{f}(\lambda)dc_0^\beta(\lambda) = c_0^\beta\hat{f}(0),$$

where c_0^β is given by Eq. (18). In other words $\lim_{k\to 0} dc_k^\beta(\lambda) = dc_0^\beta(\lambda) = c_0^\beta\delta(\lambda)d\lambda$.

Proof. The statement that $\lim_{k\to 0} dc_k^\beta(\lambda) = dc_0^\beta(\lambda)$ follows from Assumption 2. The proof of the second statement is based on the time invariance of

$$c_0^\beta(t) = \lim_{n\to\infty} \beta\big(F_{n,0}(q), \alpha_t F_{n,0}(q)\big)_\sim \quad \text{(Proposition 6)}$$

and by (21): for $\hat{f} \in \mathcal{D}$,

$$\begin{aligned}\hat{f}(\lambda)c_0^\beta &= \beta \lim_{n\to\infty} \int f(t)\big(F_{n,0}(q), \alpha_t F_{n,0}(q)\big)_\sim e^{-i\lambda t} dt \\ &= \int_0^\infty \hat{f}(\lambda - \lambda')dc_0^\beta(d\lambda'),\end{aligned}$$

i.e. $dc_0^\beta(\lambda) = c_0^\beta \delta(\lambda)d\lambda$. □

In order not to obscure the construction of the Goldstone boson normal coordinates by technical details, we will first consider the case that

$$dc_k^\beta(\lambda) = c_k^\beta \delta(\lambda - \epsilon_k^\beta)d\lambda, \tag{22}$$

with $\epsilon_k^\beta > 0$ and $c_k^\beta = \lim_{n\to\infty} \beta\big(F_{n,k}(q), \alpha_t F_{n,k}(q)\big)_\sim$. From Proposition 8 we deduce that this is a good approximation for sufficiently small $|k|$, and we will show later that this approximation becomes exact in a certain limit $k \to 0$, to be specified later.

From Eq. (21) and (22), it follows

$$\lim_{n\to\infty} \omega\Big(F_{n,k}(q)\hat{f}(H)F_{n,k}(q)\Big) = \frac{c_k^\beta \epsilon_k^\beta}{2(1 - e^{-\beta\epsilon_k^\beta})}\Big(\hat{f}(\epsilon_k^\beta) + \hat{f}(-\epsilon_k^\beta)e^{-\beta\epsilon_k^\beta}\Big). \tag{23}$$

In particular one has

$$\tilde{\omega}_k\Big(F_k(q)^2\Big) = \lim_{n\to\infty} \omega\Big(F_{n,k}(q)^2\Big) = \frac{c_k^\beta \epsilon_k^\beta}{2} \coth \frac{\beta\epsilon_k^\beta}{2}.$$

Also time invariance of $c_0^\beta(t)$ (see above) (i.e. SSB) implies

$$\lim_{k\to 0} \epsilon_k^\beta = 0, \tag{24}$$

as can be seen from (23):

$$c_0^\beta(t) = \lim_{k\to 0} c_k^\beta(t) = \lim_{k\to 0} c_k^\beta \cos \epsilon_k^\beta t.$$

For $\hat{f} \in \mathcal{D}$, denote

$$q(f) = \int f(t)\alpha_{-t} q = \hat{f}(L)q$$

and consider the equivalence class $[q(f)]_k$. For $q(f) \in \mathcal{A}_{L,sa}$ the fluctuation operator $F_k\big([q(f)]_k\big)$ is well defined,

$$\tilde{\omega}_k\Big(F_k\big([q(f)]_k\big)^2\Big) = \big\langle [q(f)]_k, [q(f)]_k\big\rangle_k = |\hat{f}(\epsilon_k^\beta)|^2 \frac{c_k^\beta \epsilon_k^\beta}{2} \coth \frac{\beta\epsilon_k^\beta}{2}, \tag{25}$$

(we used that $q(f) \in \mathcal{A}_{L,sa}$ iff $\bar{\hat{f}}(\lambda) = \hat{f}(-\lambda)$), and obviously for these functions f, we can define elements $[q]_k(f) \in \mathcal{K}_k^{\mathrm{Re}}$ through the relation $[q]_k(f) = [q(f)]_k$. However since $\mathcal{K}_k$ is by definition closed for the $\langle\cdot,\cdot\rangle_k$ topology, we can define elements $[q]_k(f)$ for a much wider class of functions $\mathcal{F}$, namely all those functions for which $|\hat{f}(\epsilon_k^\beta)| < \infty$: let f_i be a sequence of functions such that $[q(f_i)]_k \in \mathcal{K}_k^{\mathrm{Re}}$ and $\lim_i \hat{f}_i(\epsilon_k^\beta) = \hat{f}(\epsilon_k^\beta)$, and define

$$[q]_k(f) = \text{strong-}\lim_i [q(f_i)]_k.$$

In particular we have

$$[q]_k(g) \in \mathcal{K}_k^{\mathrm{Re}} \text{ with } \hat{g}(\lambda) = \frac{i}{\lambda},$$

and obviously we interpret $F_k\big([q]_k(g)\big)$ as "$F_k\big(iL^{-1}(q)\big)$", i.e. as the k-fluctuation operator of the canonical order parameter, even though $iL^{-1}(q)$ does not exist in general.

In the spirit of Proposition 4, denote

$$Q_k = F_k(q), \quad P_k = F_k\big([q]_k(g)\big) \text{ with } \hat{g}(\lambda) = \frac{i}{\lambda},$$

and denote by $\tilde{\mathcal{B}}_k$ the algebra generated by Q_k and P_k. Also denote by $\tilde{\mathcal{C}}_k$ the algebra generated by the operators $F_k\big([q]_k(f)\big)$ with $f \in \mathcal{F}$. Our main result is then that the pair (Q_k, P_k), constructed directly from the generator of the broken symmetry, forms a harmonic normal mode, therefore properly called the *Goldstone boson normal mode*. This result is an extension of Proposition 4 to the case of $k \neq 0$ fluctuations in the presence of SSB.

Theorem 9. *In the presence of SSB (Assumption 1), and in the case (22), the generator of the broken symmetry determines uniquely the construction of a canonical pair of fluctuation operators* (Q_k, P_k),

$$[Q_k, P_k] = ic_k^\beta$$

with $c_k^\beta = \lim_{n\to\infty} \beta\Big(F_{n,k}(q), F_{n,k}(q)\Big)_\sim > 0$, *satisfying a* **virial theorem**:

$$\tilde{\omega}_k\left(Q_k^2\right) = (\epsilon_k^\beta)^2 \tilde{\omega}_k\left(P_k^2\right).$$

The microscopic time evolution α_t *induces a time evolution* $\tilde{\alpha}_t^k$ *on* $\tilde{\mathcal{C}}_k$ *through the relation*

$$\tilde{\alpha}_t^k F_k\big([q]_k(f)\big) \equiv F_k\big([q]_k(U_t f)\big), \quad (\widehat{U_t f})(\lambda) = e^{it\lambda}\hat{f}(\lambda);$$

$\tilde{\alpha}_t^k$ *leaves* $\tilde{\mathcal{B}}_k$ *invariant and leads to the equations of motion*

$$\tilde{\alpha}_t^k Q_k = Q_k \cos\epsilon_k^\beta t + \epsilon_k^\beta P_k \sin\epsilon_k^\beta t, \tag{26}$$

$$\tilde{\alpha}_t^k P_k = -\frac{Q_k}{\epsilon_k^\beta}\sin\epsilon_k^\beta t + P_k\cos\epsilon_k^\beta t. \tag{27}$$

The operators (Q_k, P_k) *are called the* **Goldstone boson normal coordinates**. *The Goldstone boson creation and annihilation operators are defined by*

$$a_k^{\pm} = \frac{Q_k \mp i\epsilon_k^\beta P_k}{\sqrt{2\epsilon_k^\beta}}$$

satisfying $[a_k^-, a_k^+] = c_k^\beta$. *The quasi-free state* $\tilde{\omega}_k$ *is a* β*-KMS state on* $\tilde{\mathcal{B}}_k$ *for the evolution* $\tilde{\alpha}_t^k$, *i.e. the Goldstone bosons have a Bose–Einstein distribution:*

$$\tilde{\omega}_k\left(a_k^+ a_k^-\right) = \frac{c_k^\beta}{e^{\beta\epsilon_k^\beta} - 1},$$

which is equivalent to

$$\tilde{\omega}_k\left(Q_k^2\right) = \frac{c_k^\beta \epsilon_k^\beta}{2} \coth\frac{\beta\epsilon_k^\beta}{2}.$$

The state $\tilde{\omega}_k$ *is gauge invariant:* $\tilde{\omega}_k\left(a_k^+ a_k^+\right) = 0 = \tilde{\omega}_k\left(a_k^+\right)$.

Proof. The commutator follows from

$$\sigma_k\big([q]_k, [q]_k(g)\big) = -i\int \hat{g}(\lambda)(1 - e^{-\beta\lambda})d\tilde{\mu}_k(\lambda).$$

The variance of P_k is obtained from (25):

$$\tilde{\omega}_k\big(P_k^2\big) = \frac{c_k^\beta}{2\epsilon_k^\beta}\coth\frac{\beta\epsilon_k^\beta}{2} = \frac{1}{(\epsilon_k^\beta)^2}\tilde{\omega}_k\big(Q_k^2\big).$$

Denote $\hat{h}(\lambda) = i\lambda$. Clearly the infinitesimal generator of $\tilde{\alpha}_t^k$ is given by

$$\frac{d}{dt}\tilde{\alpha}_t^k\big|_{t=0} F_k\big([q]_k(f)\big) = F_k\big([q]_k(hf)\big).$$

Hence the first relation

$$\frac{d}{dt}\tilde{\alpha}_t^k\Big|_{t=0} P_k = -Q_k \tag{28}$$

follows trivially. The second,

$$\frac{d}{dt}\tilde{\alpha}_t^k\Big|_{t=0} Q_k = (\epsilon_k^\beta)^2 P_k, \tag{29}$$

follows from the equivalence relation (20): from Eq. (23) one computes straightforwardly

$$\big\langle [q]_k(h) - (\epsilon_k^\beta)^2[q]_k(g), [q]_k(h) - (\epsilon_k^\beta)^2[q]_k(g)\big\rangle_k = 0,$$

where $\hat{g}(\lambda) = i\lambda^{-1}$ as before. Exponentiation of (28) and (29) leads to the equations of motion. Also the remainder of the theorem follows from (23). □

Remark that for $k \to 0$, $\tilde{\omega}_k\left(P_k^2\right)$ diverges as $(\epsilon_k^\beta)^{-2}$. This divergence corresponds to the well known phenomenon of long range correlations in the order parameter fluctuations.

Similarly to what we did after Proposition 4, the proper generalisation of (22), is to consider the case that for $k \neq 0$, the support Δ_k of the measure $d\tilde{\mu}_k(\lambda)$ is bounded away from 0 and absolutely continuous, i.e.

Assumption 3. *By translation invariance we assume that for $k \neq 0$, there exists $\epsilon_k^\beta > 0$ such that $\Delta_k^+ \equiv \Delta_k \cap \mathbb{R}^+ \subseteq [\epsilon_k^\beta, +\infty)$ and that there exists a function $c_k^\beta(\lambda)$ such that*

$$dc_k^\beta(\lambda) = c_k^\beta(\lambda)d\lambda. \tag{30}$$

Equation (23) becomes

$$\lim_{n\to\infty} \omega\left(F_{n,k}(q)\hat{f}(H)F_{n,k}(q)\right) = \int_{\epsilon_k^\beta}^\infty \frac{c_k^\beta(\lambda)\lambda}{2(1-e^\beta\lambda)}\left(\hat{f}(\lambda)+\hat{f}(-\lambda)e^{-\beta\lambda}\right).$$

It is clear that again the single mode (Q_k, P_k) gets replaced by a continuous family of modes $\left\{\left(Q_k(\lambda), P_k(\lambda)\right) \mid \lambda \in \Delta_k^+\right\}$, such that

$$\begin{aligned}
\left[Q_k(\lambda), P_k(\lambda')\right] &= c_k^\beta(\lambda)\delta(\lambda-\lambda'),\\
\tilde{\omega}_k\left(Q_k(\lambda)^2\right) &= \lambda^2\tilde{\omega}_k\left(P_k(\lambda)^2\right) = \frac{c_k^\beta(\lambda)\lambda}{2}\coth\beta\lambda 2,\\
\tilde{\alpha}_t^k Q_k(\lambda) &= Q_k(\lambda)\cos\lambda t + \lambda P_k(\lambda)\sin\lambda t,\\
\tilde{\alpha}_t^k P_k(\lambda) &= -\frac{Q_k(\lambda)}{\lambda}\sin\lambda t + P_k(\lambda)\cos\lambda t,
\end{aligned}$$

and

$$Q_k = \int_{\epsilon_k^\beta}^\infty Q_k(\lambda)d\lambda, \quad P_k = \int_{\epsilon_k^\beta}^\infty P_k(\lambda)d\lambda.$$

4.4. Goldstone mode for infinite wavelength. Next we look for the Goldstone mode operators in the limit of k tending to zero, i.e. in the long wavelength limit. We take the results of Sect. 4.3 and study the limit $k \to 0$. Among other results, we show that the long wavelength Goldstone mode survives in this limit only in the ground state. This shows also that no long wavelength quantum Goldstone modes are present for temperatures $T > 0$. For $T > 0$, the spontaneous symmetry breakdown does not show any quantum behaviour, only classical modes are present.

For simplicity we will first consider the case of a single harmonic mode (Q_k, P_k), i.e. the case (22). However we will prove afterwards that the results we obtain in the limit $k \to 0$ are *independent* of this choice and are valid in general.

Let $\epsilon_k = \lim_{\beta\to\infty}\epsilon_k^\beta$, the ground state spectrum. Because of the Goldstone theorem, we have that $\lim_{k\to 0}\epsilon_k = 0$. Let $c_0^\beta = \lim_{k\to 0} c_k^\beta$ and $c_k = \lim_{\beta\to\infty} c_k^\beta$.

Assumption 4. *Assume* $\lim_{k\to 0} c_k = \lim_{\beta\to\infty} c_0^\beta = c_0 < \infty$.

First let $\beta < \infty$. The variances

$$\tilde{\omega}_k\left(Q_k^2\right) = \frac{c_k^\beta \epsilon_k^\beta}{2} \coth \frac{\beta \epsilon_k^\beta}{2} = (\epsilon_k^\beta)^2 \tilde{\omega}_k\left(P_k^2\right)$$

behave as follows for $k \to 0$:

$$\tilde{\omega}_k\left(Q_k^2\right) \approx \frac{c_k^\beta}{\beta} \to \frac{c_0^\beta}{\beta} \text{ (finite)}, \quad \tilde{\omega}_k\left(P_k^2\right) \approx \frac{c_k^\beta}{\beta(\epsilon_k^\beta)^2} \to \infty.$$

Since observable fluctuation operators are always characterized by a finite, non-zero variance, it is clear that we have to renormalize P_k before taking a limit $k \to 0$:

$$\check{P}_k = \epsilon_k^\beta P_k.$$

This however implies that the commutator

$$[Q_k, \check{P}_k] = i c_k^\beta \epsilon_k^\beta$$

vanishes in the limit $k \to 0$. In other words the quantum character and hence also the harmonic oscillation of the Goldstone mode disappears in the appropriate limit $k \to 0$, at least at non-zero temperature.

At zero temperature ($\beta = \infty$), in the ground state, the situation is completely different. The variances behave now for $k \to 0$ as follows:

$$\tilde{\omega}_k\left(Q_k^2\right) = \frac{c_k \epsilon_k}{2} \to 0, \quad \tilde{\omega}_k\left(P_k^2\right) = \frac{c_k}{2\epsilon_k} \to \infty,$$

but their product

$$\tilde{\omega}_k\left(Q_k^2\right) \tilde{\omega}_k\left(P_k^2\right) = \frac{c_k^2}{4} \to \frac{c_0^2}{4}$$

remains finite. This means that the divergence of the order parameter operator fluctuations due to long range correlations is exactly compensated by a proportional squeezing of the symmetry generator fluctuations. Therefore one can find a renormalized Q_k and P_k, denoted by $\check{Q}_k$ and $\check{P}_k$, having both a finite, non-zero variance, with a finite non-zero commutator; indeed take e.g.

$$\check{Q}_k \equiv \epsilon_k^{-1/2} Q_k, \quad \check{P}_k \equiv \epsilon_k^{1/2} P_k,$$

then

$$\tilde{\omega}_k\left(\check{Q}_k^2\right) = \tilde{\omega}_k\left(\check{P}_k^2\right) = \frac{c_k}{2} \to \frac{c}{2}, \quad [\check{Q}_k, \check{P}_k] = i c_k \to i c.$$

Remark that this scaling transformation has no effect on the creation and annihilation operators, in particular:

$$a_k^\pm = \frac{Q_k \mp i \epsilon_k P_k}{\sqrt{2\epsilon_k}} = \frac{\check{Q}_k \mp i \check{P}_k}{\sqrt{2}}.$$

On the other hand, the equations of motion (26) and (27) are transformed into

$$\tilde{\alpha}_t^k \check{Q}_k = \check{Q}_k \cos \epsilon_k t + \check{P}_k \sin \epsilon_k t,$$
$$\tilde{\alpha}_t^k \check{P}_k = -\check{Q}_k \sin \epsilon_k t + \check{P}_k \cos \epsilon_k t.$$

Hence in order to retain a non-trivial time evolution in the $k \to 0$ limit, one has to rescale time as well in the following way: $t \to \tau = \epsilon_k t$.

Let $\tilde{\mathcal{B}}_0$ be an algebra generated by a canonical pair $(\check{Q}_0, \check{P}_0)$,

$$\left[\check{Q}_0, \check{P}_0\right] = i c_0;$$

$\tilde{\alpha}_\tau^0$, $\tau \in \mathbb{R}$ is a time evolution on $\tilde{\mathcal{B}}_0$ defined through the equations of motion

$$\tilde{\alpha}_\tau^0 \check{Q}_0 = \check{Q}_0 \cos \tau + \check{P}_0 \sin \tau,$$
$$\tilde{\alpha}_\tau^0 \check{P}_0 = -\check{Q}_0 \sin \tau + \check{P}_0 \cos \tau,$$

and $\tilde{\omega}_0$ is a state on $\tilde{\mathcal{B}}_0$ defined through the relation

$$\tilde{\omega}_0\big(F(\check{Q}_0, \check{P}_0)\big) \equiv \lim_{k \to 0} \tilde{\omega}_k\big(F(\check{Q}_k, \check{P}_k)\big),$$

where F is any polynomial in two variables. Summarizing our results:

Theorem 10. *In the ground state ($\beta = \infty$), the dynamical system $(\tilde{\mathcal{B}}_k, \tilde{\alpha}_t^k, \tilde{\omega}_k)$ converges in the limit $k \to 0$ to the dynamical system $(\tilde{\mathcal{B}}_0, \tilde{\alpha}_\tau^0, \tilde{\omega}_0)$ in the sense that for any two polynomials F_1, F_2 in two variables,*

$$\tilde{\omega}_0\big(F_1(\check{Q}_0, \check{P}_0)\tilde{\alpha}_\tau^0 F_2(\check{Q}_0, \check{P}_0)\big) = \lim_{k \to 0} \tilde{\omega}_k\big(F_1(\check{Q}_k, \check{P}_k)\tilde{\alpha}^k_{\frac{\tau}{\epsilon_k}} F_2(\check{Q}_k, \check{P}_k)\big).$$

Therefore we can identify $\check{Q}_0 = \lim_{k \to 0} \check{Q}_k$ and $\check{P}_0 = \lim_{k \to 0} \check{P}_k$. Moreover $\tilde{\omega}_0$ is a ground state for $\tilde{\alpha}_\tau^0$, i.e. for all $X \in \tilde{\mathcal{B}}_0$,

$$\frac{d}{dt}\bigg|_{t=0} \tilde{\omega}_0\big(X^* \tilde{\alpha}_\tau^0 X\big) \geq 0.$$

The pair $(\check{Q}_0, \check{P}_0)$ is called **the canonical pair of the collective Goldstone mode**.

Proof. Due to quasi-freeness, it is sufficient to check these properties for the two-point correlation function. But in this case they follow immediately from the very definition of $\tilde{\alpha}_\tau^0$ and $\tilde{\omega}_0$. □

Remark that although formally, Theorem 9 and 10 are very similar, it is important to remember the rescaling that has been done. In fact the previous theorem tells us that in the ground state the long range correlations in the order parameter fluctuations are exactly compensated by a squeezing of the generator fluctuations. Both operators continue to form a harmonic oscillator pair in the limit $k \to 0$, although the frequency becomes infinitesimally small and hence the period of oscillation infinitely (or macroscopically) large.

Considering the most common case of powerlaw behaviour of the energy spectrum, i.e. $\epsilon_k = \epsilon |k|^\delta$, this rescaling provides information about the size of the 0-mode fluctuations. In a finite box Λ_n of length $L = 2n + 1$, the smallest non-zero wave vector has length $|k| \propto L^{-1}$. Therefore the rescaling of Q_k with a factor $\epsilon_k^{-1/2}$ suggests a rescaling by $L^{\delta/2} = |\Lambda_n|^{\delta/2\nu}$ of the fluctuation, i.e.

$$F_{n,0}(q) = \frac{1}{|\Lambda_n|^{\frac{1}{2}-\frac{\delta}{2\nu}}} \sum_{x\in\Lambda_n} \big(q_x - \omega(q)\big),$$

in order that its variance is non-zero and finite. This means that the fluctuations of the symmetry generator are of order $|\Lambda_n|^{\frac{1}{2}-\frac{\delta}{2\nu}}$, i.e. *subnormal fluctuations*. Similarly the fluctuations of the order parameter are of order $|\Lambda_n|^{\frac{1}{2}+\frac{\delta}{2\nu}}$, i.e. *abnormal fluctuations*. This requires $\frac{\delta}{2\nu} \leq \frac{1}{2}$, or $\delta \leq \nu$. This condition is undoubtly related to the condition $c < \infty$ (Assumption 4). Remark also that if SSB disappears, i.e. if $c = 0$, then the Goldstone boson disappears.

Finally we remark that the results of Theorem 10 do not depend on the particular form of the measure $dc_k^\beta(\lambda)$, in this case given by (22). One could equally well take the more general form (30), since in the limit $k \to 0$ this measure also reduces to a δ-peak by Proposition 8. It is a straightforward calculation to show that Theorem 10 holds in general (i.e. under Assumption 3), upon interpreting ϵ_k as the gap in the support of the measure $dc_k(\lambda)$.

Therefore we find that at zero temperature, the fluctuations of the symmetry generator lead to a single harmonic mode with vanishingly small frequency in the long-wavelength limit, even though at finite wavelength, there exists a continuous family of modes associated to the fluctuations of the symmetry generator. It is hence also appropriate to consider the results of Theorem 9 as being physically valid in general, as long as one considers low enough temperatures and large enough wavelengths.

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Communicated by H. Araki